Instantons and Kaehler Geometry of Nilpotent Orbits

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Abstract

The first obstacle in building a Geometric Quantization theory for nilpotent orbits of a real semisimple Lie group has been the lack of an invariant polarization. In order to generalize the Fock space construction of the quantum mechanical oscillator, a polarization of the symplectic orbit invariant under the maximal compact subgroup is required.

In this paper, we explain how such a polarization on the orbit arises naturally from the work of Kronheimer and Vergne. This occurs in the context of hyperkaehler geometry. The polarization is complex and in fact makes the orbit into a (positive) Kaehler manifold. We study the geometry of this Kaehler structure, the Vergne diffeomorphism, and the Hamiltonian functions giving the symmetry. We indicate how all this fits into a quantization program.

1   Introduction

Quantization is a procedure for constructing a quantum system with symmetry out of a classical system with symmetry. No axiomatic or even systematic method of quantization is known. Instead, quantization exists as an empirical science, made up of a growing series of examples. In many ways, quantization is an art.

The nature of the classical and quantum systems under consideration depends on the context and on the scope of the investigation. At present a universal sort of quantization scheme seems completely out of reach. Such a scheme would have to include the quantization of gravity as well as the quantization of classical field theory. In fact, some physicists believe that even familiar classical theories must be modified in order that they can be "quantized" to give a consistent and meaningful quantum theory.

There is a rather clear "beginning level" at which to formulate and study the quantization problem. This is the case where one starts with a classical Hamiltonian mechanical (dynamical) system with symmetry. Such a system is given by a phase space (M, ω) together with a Hamiltonian function F and a Lie subalgebra

$$\mathfrak{g} \subset C^\infty(M) \tag{1.1}$$

Here (M, ω) is a real symplectic manifold; i.e., ω is a symplectic form on a smooth manifold M of dimension $2n$. The symplectic structure ω defines a Poisson bracket $\{ \, , \, \}$ on $C^\infty(M)$, giving it the structure of a Poisson algebra. The Hamiltonian is a fixed smooth function $F : M \to \mathbb{R}$ which determines the time evolution (the dynamics) of the system. Each smooth function ϕ on M determines a Hamiltonian vector field ξ_ϕ by the formula $\xi_\phi \lrcorner \omega + d\phi = 0$. The Poisson bracket is given by $\{\phi, \psi\} = \xi_\phi(\psi) = \omega(\xi_\phi, \xi_\psi)$ and satisfies

$$[\xi_\phi, \xi_\psi] = \xi_{\{\phi, \psi\}} \tag{1.2}$$

Thus the Hamiltonian vector fields ξ_ϕ of the functions $\phi \in \mathfrak{g}$ give an infinitesimal Lie algebra action of $\mathfrak{g}$ on M. This constitutes *infinitesimal Hamiltonian symmetry*. If the $\mathfrak{g}$-action integrates to a smooth action of a Lie group G on M, then this G-action is called Hamiltonian. Regardless of integration, the inclusion $\mathfrak{g} \hookrightarrow C^\infty(M)$ defines a smooth infinitesimally $\mathfrak{g}$-equivariant *moment map*

$$M \to \mathfrak{g}^* \tag{1.3}$$

If G-acts transitively on M in a Hamiltonian fashion, then the moment map (1.3) is just a covering onto a coadjoint orbit of G. This focuses attention on coadjoint orbits as the "elementary Hamiltonian spaces" (up to covering). Moreover, each coadjoint orbit P has a canonical symplectic structure σ, sometimes called the KKS (Kirillov-Kostant-Souriau) symplectic structure, derived from the Lie algebra bracket on $\mathfrak{g}$. Indeed, each $x \in \mathfrak{g}$ defines a linear function ϕ^x on $\mathfrak{g}^*$ and hence on P. Then σ is the unique symplectic form such that the mapping

$$\mathfrak{g} \to C^\infty(P), \qquad x \mapsto \phi^x \tag{1.4}$$

preserves brackets, i.e., is a Lie algebra homomorphism. This discussion applies equally well in the category of holomorphic symplectic manifolds; cf. [BK1].

An outstanding problem is the quantization of conical coadjoint orbits $\mathcal{O}_\mathbb{R}$ of a real semisimple Lie group $G_\mathbb{R}$. Here we take $G_\mathbb{R}$ to be a real form of a connected and simply-connected complex semisimple Lie group G with Lie algebra $\mathfrak{g}$. We may identify $\mathfrak{g}_\mathbb{R} = Lie G_\mathbb{R}$ with its dual $\mathfrak{g}_\mathbb{R}^*$, and then the conical orbits $\mathcal{O}_\mathbb{R}$ identify with the orbits of nilpotent elements in $\mathfrak{g}_\mathbb{R}$. These are the so-called "nilpotent orbits". The irreducible unitary representations arising from quantization of nilpotent orbits are often called "unipotent" representations. These are examples of "singular representations".

In order to start a quantization program for nilpotent orbits, we need at the outset an invariant polarization of $\mathcal{O}_\mathbb{R}$. In analogy with the well-known quantization of the harmonic oscillator, we want a polarization invariant under a maximal compact subgroup $K_\mathbb{R}$ of $G_\mathbb{R}$. (In general $\mathcal{O}_\mathbb{R}$ does not admit $G_\mathbb{R}$-invariant polarizations.)

Remarkably, a $K_\mathbb{R}$-invariant polarization arises, in a uniform natural manner on every real nilpotent orbit, from the work of Kronheimer and Vergne. This comes about by first working on the complexification $\mathcal{O}$ of $\mathcal{O}_\mathbb{R}$; $\mathcal{O} \subset \mathfrak{g}$ is a complex nilpotent orbit of G. We let $\mathbf{I}$ denote the natural complex structure on $\mathcal{O}$; then $\mathcal{O}_\mathbb{R}$ is an $\mathbf{I}$-real form.

Kronheimer ([Kr]) in 1990 identified each complex nilpotent orbit O as an instanton moduli space. In particular the holomorphic symplectic structure $(\mathbf{I}, \Sigma)$ on O extends to a hyperkaehler structure $(g, \mathbf{I}, \mathbf{J}, \mathbf{K}, \omega_\mathbf{I}, \omega_\mathbf{J}, \omega_\mathbf{K})$ where $\Sigma = \omega_\mathbf{J} + i\omega_\mathbf{K}$. We outline the Kronheimer model of $\mathcal{O}$ in §5.

Then Vergne in 1995 used this to discover a diffeomorphism

$$\mathcal{V} : \mathcal{O}_{\mathbb{R}} \to Y \tag{1.5}$$

of each real nilpotent orbit $\mathcal{O}_{\mathbb{R}}$ to a complex K-homogeneous cone Y, where $K \subset G$ is the complexification of $K_{\mathbb{R}}$. This recovered the Kostant-Sekiguchi ([Se]) correspondence.

The upshot of Vergne's work on the Kronheimer instanton model of $\mathcal{O}$ is that $\mathcal{O}_{\mathbb{R}}$ is a **J**-complex submanifold of $\mathcal{O}_{\mathbb{R}}$. Moreover, $\mathcal{O}_{\mathbb{R}}$ is then a Kaehler submanifold of $\mathcal{O}$ with respect to $(\mathbf{J}, \omega_{\mathbf{J}})$. But then $\omega_{\mathbf{J}} = \mathrm{Re}\,\Sigma$ is just the real KKS symplectic form on $\mathcal{O}_{\mathbb{R}}$.

So the "new" complex structure **J** provides a complex polarization on $\mathcal{O}_{\mathbb{R}}$ and moreover makes $\mathcal{O}_{\mathbb{R}}$ into a Kaehler manifold which identifies with Y as a complex manifold. We explain the Vergne theory and the Kaehler structure in §6 and §7. In §6 we also give a different proof of Vergne's result (see especially Proposition 6.6 and Corollaries 6.5 and 6.8).

In §7,8,9 we develop the properties of the Vergne diffeomorphism and the Kaehler structure. Our first main result is the Triple Sum formula in in Theorem 7.9. This leads to our key result for quantization in Theorem 9.3 on the nature of the Hamiltonian functions ϕ^x. Indeed we have the Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$. While the Hamiltonian flows of the functions ϕ^x, $x \in \mathfrak{k}_{\mathbb{R}}$, preserve **J**, the flows of the remaining functions ϕ^v, $v \in \mathfrak{p}_{\mathbb{R}}$, do not preserve **J**. The question then is how will they quantize. On a classical level, we can ask how to write down ϕ^v in terms of holomorphic and antiholomorphic functions. The answer in Theorem 9.3 is that ϕ^v is the real part of a holomorphic function. The interpretation of this is discussed further in §9.

In §8, we explain another aspect of the Kaehler structure, namely that there is a global Kaehler potential ρ_o on $(\mathcal{O}_{\mathbb{R}}, \mathbf{J}, \sigma)$. This function $\rho_o : \mathcal{O}_{\mathbb{R}} \to \mathbb{R}$ is $K_{\mathbb{R}}$-invariant and uniquely determined by the condition that it is homogeneous of degree 1 under the Euler scaling action of $\mathbb{R}^+$.

This Kaehler potential arises by restriction from the hyperkaehler potential on $\mathcal{O}$. In §2,§3,§4 we develop the basic theory of hyperkaehler manifolds, hyperkaehler cones and hyperkaehler potentials based on results from [HKLR] and [Sw].

The importance of the Kaehler potential ρ_o is this: in our quantization program for real nilpotent orbits, ρ_o plays the role of the Hamiltonian, i.e., the energy function. Moreover, ρ_o gives rise in Theorem 8.6 and Corollary 8.7 to a realization of T^*Y as a holomorphic symplectic complexification of $\mathcal{O}_{\mathbb{R}}$.

Our quantization program building on this geometry will be developed in subsequent papers. See also [B1],[B2], [BK2].

In the quantization of $(\mathcal{O}_{\mathbb{R}}, \mathbf{J}, \sigma)$, we want to "quantize" the Hamiltonian functions ϕ^z, $z \in \mathfrak{g}_{\mathbb{R}}$, by converting the ϕ^z into self-adjoint operators $\mathcal{Q}(\phi^z)$ on a space of holomorphic half-forms on $(\mathcal{O}_{\mathbb{R}}, \mathbf{J}) \cong Y$. The conversion must satisfy in particular Dirac's axiom that Poisson bracket of functions goes over into the commutator of operators so that $\mathcal{Q}(\{\phi^z, \phi^w\}) = i[\mathcal{Q}(\phi^z), \mathcal{Q}(\phi^w)]$.

A main idea coming out of Corollary 8.7 is that we can try to "promote" the functions ϕ^z on $\mathcal{O}_{\mathbb{R}}$ to rational functions on the holomorphic symplectic complexification T^*Y. For this to work, we need some sort of analyticity and algebraicity for the embedding of $\mathcal{O}_{\mathbb{R}}$ into T^*Y.

The appropriate notion combining analyticity and algebraicity here turns out to be that of a Nash embedding. In the Appendix, we give an outline of Nash geometry, starting from the theory of real algebraic varieties. O. Biquard has proven in [Bi] that the hyperkaehler potential on $\mathcal{O}$, and hence the $SO(3)$-action on $\mathcal{O}$ and Vergne diffeomorphism (1.5), are Nash.

2   Hyperkaehler Manifolds

In this section, we review and perhaps clarify some basic notions of hyperkaehler geometry that we use throughout this paper.

A hyperkaehler manifold $(X, g, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$ is real manifold X of dimension $4n$ together with a Riemannian metric g and three complex structures $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ such that (i) $\mathbf{J}_1 \mathbf{J}_2 \mathbf{J}_3 = -1$ and (ii) g is a Kaehler metric with respect to each of $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$.

Then by (i), $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ satisfy the quaternion relations

$$\mathbf{J}_1^2 = \mathbf{J}_2^2 = \mathbf{J}_3^2 = -1, \qquad \mathbf{J}_a \mathbf{J}_b = \varepsilon_{abc} \mathbf{J}_c$$

Here $a, b, c \in \{1, 2, 3\}$ are distinct and $\varepsilon_{abc} = \mathrm{sgn}(abc)$. Thus every tangent space of X becomes a quaternionic vector space.

By (ii), X has three Kaehler manifold structures $(\mathbf{J}_1, \omega_1)$, $(\mathbf{J}_2, \omega_2)$, $(\mathbf{J}_3, \omega_3)$, all with Kaehler metric g. The Kaehler forms ω_a are given by $g(u, v) = \omega_a(u, \mathbf{J}_a v)$. We call these Kaehler manifolds X_1, X_2, X_3, respectively.

The data $(X, g, \omega_1, \omega_2, \omega_3)$ serves equally well to define the hyperkaehler structure as we may recover the complex structures by the formula

$$\omega_c(u, v) = \omega_a(\mathbf{J}_b u, v) \varepsilon_{abc} \tag{2.1}$$

We define three complex 2-forms on X

$$\Omega_1 = \omega_2 + i\omega_3, \quad \Omega_2 = \omega_3 + i\omega_1, \quad \Omega_3 = \omega_1 + i\omega_2 \tag{2.2}$$

Then Ω_a is $\mathbf{J}_a$-holomorphic. This is shown in [HKLR, pp. 549-550].

Inside the quaternion algebra

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$$

we have the standard 2-sphere

$$S^2 = \{q = a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \,|\, |q| = 1\}$$

of pure imaginary quaternions of unit norm.

Corresponding to a point $q = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ on S^2, we have the pair

$$\mathbf{J}_q = a\mathbf{J}_1 + b\mathbf{J}_2 + c\mathbf{J}_3 \qquad \text{and} \qquad \omega_q = a\omega_1 + b\omega_2 + c\omega_3$$

Then $(X, g, \mathbf{J}_q, \omega_q)$ is again a Kaehler structure on X with complex structure $\mathbf{J}_q$ and Kaehler form ω_q; we write X_q for this Kaehler manifold. Thus we have a 2-sphere $\mathcal{S}_X$ of Kaehler structures $(\mathbf{J}_q, \omega_q)$ on X and we have identified $\mathcal{S}_X$ with S^2.

Let $q \mapsto \tau q$ be the standard rotation action of $SO(3)$ on S^2. This induces an $SO(3)$-action on $\mathcal{S}_X$ given by $\tau \cdot \mathbf{J}_q = \mathbf{J}_{\tau q}$ and $\tau \cdot \omega_q = \omega_{\tau q}$. Let

$$\mathcal{C}_q \subset SO(3) \tag{2.3}$$

be the circle subgroup of which fixes $q \in S^2$.

The generalization of (2.2) is that any $q' \in S^2$ orthogonal to q determines a $\mathbf{J}_q$-holomorphic symplectic form $\omega_{q'} + i\omega_{q''}$ on X where $q'' = q \times q'$ is the cross product of q with q'.

Example 2.1 The first example of a hyperkaehler manifold is the flat quaternionic vector space. Let $X = \mathbb{R}^{4n}$ with standard linear coordinates x_s^r where $r = 0, 1, 2, 3$ and $s = 1, \ldots, n$. We may make X into an n-dimensional quaternionic vector space, where $\mathbb{H}$ acts by left multiplication, in the obvious way so that the functions

$$q_s = x_s^0 + x_s^1 \mathbf{i} + x_s^2 \mathbf{j} + x_s^3 \mathbf{k} \tag{2.4}$$

are quaternionic linear coordinates.

The following data defines a hyperkaehler structure on X: left multiplication by $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ give the complex structures $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ so that

$$\mathbf{J}_a \left(\frac{\partial}{\partial x_s^0} \right) = \frac{\partial}{\partial x_s^a}, \qquad \mathbf{J}_a \left(\frac{\partial}{\partial x_s^b} \right) = \frac{\partial}{\partial x_s^c} \tag{2.5}$$

where (abc) is a cyclic permutation of $1, 2, 3$. Under $\mathbf{J}_a$, X identifies with $\mathbb{C}^{2n}$ with linear holomorphic coordinate functions $x_s^0 + ix_s^a$, $x_s^b + ix_s^c$. The hyperkaehler metric is

$$g = \sum_{r,s} (\mathrm{d}x_s^r)^2 \tag{2.6}$$

The three Kaehler forms $\omega_1, \omega_2, \omega_3$ are

$$\omega_a = \sum_{s=1}^{n} \mathrm{d}x_s^0 \wedge \mathrm{d}x_s^a + \mathrm{d}x_s^b \wedge \mathrm{d}x_s^c \tag{2.7}$$

Next we introduce hyperkaehler symmetry into the picture. Let U be a Lie group. A *hyperkaehler action* of U on $(X, g, \mathbf{J}_q, \omega_q)$ is a smooth Lie group action of U on X which preserves all the hyperkaehler structure.

From now on we assume that U is a compact connected semisimple Lie group and we have a hyperkaehler action of U on X. Then differentiation gives an infinitesimal action of the Lie algebra $\mathfrak{u}$ of U by the vector fields ξ^u where $\xi_p^u = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(\exp -tu) \cdot p$ at $p \in X$. In other words, we get a Lie algebra homomorphism

$$\mathfrak{u} \to \mathfrak{Vect}\, X, \qquad u \mapsto \xi^u \tag{2.8}$$

Now consider each Kaehler manifold X_q. We let $C^\infty(X)_{\omega_q}$ denote the algebra $C^\infty(X)$ equipped with the Poisson bracket defined by ω_q.

The U-action on X is symplectic with respect to ω_q and consequently, since $\mathfrak{u}$ is semisimple, is Hamiltonian. This means that we can solve the equations $\xi^u \lrcorner \omega_q + \mathrm{d}\zeta_q^u = 0$ for functions

ζ_q^u such that $\{\zeta_q^u, \zeta_q^v\} = \zeta_q^{[u,v]}$. The momentum functions ζ_q^u are uniquely determined. So we get a Lie algebra homomorphism

$$\mathfrak{u} \to C^\infty(X)_{\omega_q}, \qquad u \mapsto \zeta_q^u \tag{2.9}$$

The corresponding U-invariant moment map

$$\zeta_q : X \to \mathfrak{u} \tag{2.10}$$

is defined by $\zeta_q^u(p) = (u, \zeta_q(p))_{\mathfrak{u}}$. Here we identify $\mathfrak{u} \simeq \mathfrak{u}^*$ by means of the Killing form $(\ ,\ )_{\mathfrak{u}}$.

Consider now the three moment maps $\zeta_1 = \zeta_{\mathbf{i}}$, $\zeta_2 = \zeta_{\mathbf{j}}$, $\zeta_3 = \zeta_{\mathbf{k}}$, Putting these together we obtain a triple moment map

$$\zeta = (\mathbf{i}\zeta_1, \mathbf{j}\zeta_2, \mathbf{k}\zeta_3) : X \to \mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u} \tag{2.11}$$

Let G be the complexification of U. Then G is the complex semisimple algebraic group characterized by either of the following properties: (i) any linear representation of U on a complex (finite-dimensional) vector space extends uniquely to a linear representation of G, or (ii) U is a compact real form of G. It follows from (ii) that U and G have the same fundamental group.

We assume now that U, and hence G, is simply-connected. The Lie algebra of G is the complex semisimple Lie algebra

$$\mathfrak{g} = \mathfrak{u} \otimes \mathbb{C} = \mathfrak{u} \oplus \mathbf{i}\mathfrak{u} \tag{2.12}$$

We identify $\mathfrak{g} \simeq \mathfrak{g}^*$ using the complex Killing form $(\ ,\ )_{\mathfrak{g}}$ on $\mathfrak{g}$. We note that $(u,v)_{\mathfrak{u}} = (u,v)_{\mathfrak{g}}$ for $u, v \in \mathfrak{u}$. This follows because an $\mathbb{R}$-linear map $L : \mathfrak{u} \to \mathfrak{u}$ determines a $\mathbb{C}$-linear map $L_{\mathbb{C}} : \mathfrak{g} \to \mathfrak{g}$ and then $Tr_{\mathbb{R}}L = Tr_{\mathbb{C}}L_{\mathbb{C}}$.

Now we consider the holomorphic symplectic manifolds $(X, \mathbf{J}_a, \Omega_a)$, $a = 1, 2, 3$. We let $R^{\mathrm{hol}}(X_a)$ denote the algebra of $\mathbf{J}_a$-holomorphic functions on X equipped with the Poisson bracket defined by Ω_a.

Since $\mathcal{L}_{\xi^u}\mathbf{J}_a = \mathbf{J}_a$, $u \in \mathfrak{u}$, it follows that $\frac{1}{2}\xi^u$ is the real part of a unique $\mathbf{J}_a$-holomorphic vector field Ξ_a^u on X. Precisely, $\frac{1}{2}\xi^u = \mathrm{Re}\,\Xi_a^u$ where

$$\Xi_a^u = \frac{1}{2}(\xi^u - i\mathbf{J}_a\xi^u) \tag{2.13}$$

Then we get the bracket relations for $u, v \in \mathfrak{u}$

$$\{\Xi_a^u, \Xi_a^v\} = \Xi_a^{[u,v]}$$

Now we have an infinitesimal $\mathbf{J}_a$-holomorphic Lie algebra action

$$\mathfrak{g} \to \mathfrak{Vect}_{\mathbf{J}_a-hol}(X), \qquad z = u + iv \mapsto \Xi_a^z = \Xi_a^u + i\Xi_a^v \tag{2.14}$$

of $\mathfrak{g}$ on X_a. This is the complexification of the infinitesimal $\mathfrak{u}$-action (2.8). If (2.13) integrates to a holomorphic G-action on $(X, \mathbf{J}_a)$, then we will say that the U-action on X *complexifies* with respect to $\mathbf{J}_a$.

Regardless of integration, the infinitesimal action (2.14) of $\mathfrak{g}$ preserves the holomorphic symplectic form Ω_a defined in (2.2). Then, since $\mathfrak{g}$ is semisimple, the infinitesimal action (2.14) is Hamiltonian. We have a unique complex Lie algebra homomorphism

$$\mathfrak{g} \to R^{\mathrm{hol}}(X_a), \qquad z \mapsto \Phi_a^z \tag{2.15}$$

given by momentum functions Φ_a^z so that $\Xi_a^z \lrcorner \Omega_a + \mathrm{d}\Phi_a^z = 0$. Then

$$\Phi_a^{u+iv} = \Phi_a^u + i\Phi_a^v \qquad \text{and} \qquad \Phi_a^u = \zeta_b^u + i\zeta_c^u \tag{2.16}$$

where (abc) is a cyclic permutation of $1, 2, 3$.

The corresponding $\mathfrak{g}$-equivariant $\mathbf{J}_a$-holomorphic moment map

$$\Phi_a : X_a \to \mathfrak{g} \tag{2.17}$$

is defined by $\Phi_a^z(p) = (z, \Phi_a(p))_{\mathfrak{g}}$. Thus we get the three maps

$$\Phi_1 = \zeta_2 + i\zeta_3, \qquad \Phi_2 = \zeta_3 + i\zeta_1, \qquad \Phi_3 = \zeta_1 + i\zeta_2 \tag{2.18}$$

The formulas (2.16)-(2.18) encode a lot of information about the coupling of the complex and symplectic structures on X, as Φ_a is $\mathbf{J}_a$-holomorphic. In particular they show how the real functions ζ_1^u, ζ_2^u, ζ_3^u give rise to holomorphic functions on X.

Example 2.2 We continue the discussion of $X = \mathbb{H}^n$ from Example 2.1. Let U be the group of all $\mathbb{R}$-linear transformations of X which preserve g and commute with the $\mathbb{H}^*$-action on X. Then U is the familiar model of the compact symplectic group $Sp(n)$. Clearly this U-action preserves all the hyperkaehler data on $\mathbb{H}^n$. In the case $n = 1$ then $U \simeq SU(2)$ and moreover U acts by right multiplication by quaternions of unit norm.

The U-action complexifies, with respect to any complex structure $\mathbf{J}_q \in \mathcal{S}_X$, to a complex linear complex algebraic action of $G \simeq Sp(2n, \mathbb{C})$ on $\mathbb{H}^n$. This action is transitive on $\mathbb{H}^n - \{0\}$.

We have a free $\mathbb{Z}_2$-action on $\mathbb{H}^n - \{0\}$ by multiplication by ± 1. This $\mathbb{Z}_2$-action preserves all the hyperkaehler data on $\mathbb{H}^n - \{0\}$ and commutes with the $SU(2)$ and G-actions. The quotient $\mathcal{O} = (\mathbb{H}^n - \{0\})/\mathbb{Z}_2$ inherits a hyperkaehler structure, an action of $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ and a U-action.

3 Hyperkaehler Cones

In this section we explain the notion of a hyperkaehler cone.

To begin with we recall that a *symplectic cone* of positive integer weight k is a symplectic manifold (M, ω) together with a smooth action

$$\mathbb{R}^+ \times M \to M, \qquad (t, m) \mapsto \gamma_t(m)$$

of the group $\mathbb{R}^+$ of positive real numbers such that

$$\gamma_t^* \omega = t^k \omega \tag{3.1}$$

This means that the $\mathbb{R}^+$-action scales the symplectic form and it has weight k.

The prototype example is the case where $k = 1$ and $M = T^*Q$ is a cotangent bundle with its canonical symplectic structure. Here $\mathbb{R}^+$ acts on T^*Q by the linear scaling action on the fibers of the projection $T^*Q \to Q$.

Let η be the infinitesimal generator of the $\mathbb{R}^+$-action. Then differentiating (3.1) we get the equivalent condition

$$\mathcal{L}_\eta \omega = k\omega \tag{3.2}$$

It follows that ω is exact with symplectic potential $\frac{1}{k}(\eta \lrcorner \omega)$; i.e.,

$$\omega = \mathrm{d}(\frac{1}{k}\eta \lrcorner \omega)$$

We conclude in particular that a symplectic cone is non-compact (and has positive dimension).

Next we define a *Kaehler cone* of weight k to be a Kaehler manifold $(Z, \mathbf{J}, \omega, g)$ together with a smooth action

$$\gamma : \mathbb{C}^* \times Z \to Z, \qquad (s, m) \mapsto \gamma_s(m) \tag{3.3}$$

which satisfies the three conditions

$$(i) \text{ the action } \gamma \text{ is holomorphic}, \qquad (ii)\gamma_s^*\omega = |s|^k\omega, \qquad (iii)\gamma_s^*g = |s|^k g \tag{3.4}$$

(These are consistent with redundancy).

The condition (i) means that the map (3.3) is holomorphic. So (i) implies $\gamma_s^*\mathbf{J} = \mathbf{J}$. Also any two of $\gamma_s^*\mathbf{J} = \mathbf{J}$, (ii), (iii) imply the other.

To work out (ii) and (iii), we use the product decomposition

$$\mathbb{C}^* = \mathbb{R}^+ \times S^1$$

So the $\mathbb{C}^*$ action splits into a product of an $\mathbb{R}^+$-action with an S^1-action. Then (ii) and (iii) say: ω and g are homogeneous of degree k under the $\mathbb{R}^+$-action, but they are fixed by the S^1-action.

Thus S^1 acts by Kaehler automorphisms. In particular, the S^1-action is symplectic and so has a moment map on Z at least locally with values in $\mathbb{R}$. We can write this moment map as $\frac{k}{2}\rho$. Then ρ is a local Kaehler potential, i.e., ρ satisfies

$$i\partial\overline{\partial}\rho = \omega \tag{3.5}$$

where $d = \partial + \overline{\partial}$ is the standard decomposition of d into $(1, 0)$ and $(0, 1)$ parts.

The Lie algebra of $\mathbb{C}^*$ is $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathbf{i}$ with $[1, \mathbf{i}] = 0$. Differentiating the $\mathbb{C}^*$-action we get an infinitesimal vector field action

$$\psi : \mathbb{C} \to \mathfrak{Vect}\, Z, \qquad v \mapsto \psi^v = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \gamma_{\exp -tv} \tag{3.6}$$

We put

$$\eta = \psi^{-1} \qquad \text{and} \qquad \theta = \psi^{-\mathbf{i}} \tag{3.7}$$

so that η and θ are, respectively, the infinitesimal generators of the actions of $\mathbb{R}^+$ and S^1. Since ψ is a (real) Lie algebra homomorphism we have

$$[\eta, \theta] = 0 \tag{3.8}$$

Notice that the infinitesimal generator of the holomorphic action of $\mathbb{C}^*$ on Z is the holomorphic vector field

$$E = \frac{1}{2}\eta - \frac{1}{2}i\theta \tag{3.9}$$

Now, we can give an equivalent infinitesimal version of the conditions (3.4) on γ: ψ must be $\mathbb{C}$-linear, i.e.,

$$\theta = \mathbf{J}\eta \tag{3.10}$$

and also

$$\mathcal{L}_\eta \mathbf{J} = 0, \quad \mathcal{L}_\eta \omega = k\omega, \quad \mathcal{L}_\eta g = kg, \quad \mathcal{L}_\theta \mathbf{J} = \mathcal{L}_\theta \omega = \mathcal{L}_\theta g = 0 \tag{3.11}$$

Notice that the condition $\mathcal{L}_\eta \mathbf{J} = 0$ itself implies $[\eta, \mathbf{J}\eta] = 0$.

Now we define a *hyperkaehler cone* of weight k to be a hyperkaehler manifold $(X, g, \mathbf{J}_q, \omega_q)$ together with a left $\mathbb{H}^*$-action

$$\gamma : \mathbb{H}^* \times X \to X, \qquad (h, m) \mapsto \gamma_h(m) \tag{3.12}$$

which satisfies the three conditions

$$
\begin{array}{lll}
(i) & \gamma_h^* \mathbf{J}_q & = \mathbf{J}_{h^{-1}qh} \text{ and the action of } \mathbb{C}_q^* \text{ on } X \text{ is } \mathbf{J}_q\text{-holomorphic} \\
(ii) & \gamma_h^* \omega_q & = |h|^k \omega_{h^{-1}qh} \\
(iii) & \gamma_h^* g & = |h|^k g
\end{array} \tag{3.13}
$$

Here $h \in \mathbb{H}^*$, $q \in S^2$ and

$$\mathbb{C}_q^* = \{a + bq \,|\, (a, b) \in \mathbb{R}^2, (a, b) \neq (0, 0)\} \tag{3.14}$$

Again (i)-(iii) are consistent with redundancies. It suffices to check (i)-(iii) just for $q = \mathbf{i}, \mathbf{j}, \mathbf{k}$.

We have the natural direct product decomposition

$$\mathbb{H}^* = \mathbb{R}^+ \times SU(2)$$

The formulation of (i)-(iii) in terms of the component actions of $\mathbb{R}^+$ and $SU(2)$ is: (a) for all q, $\mathbb{R}^+$ acts $\mathbf{J}_q$-holomorphically and scales g and ω_q so that they have weight k, (b) the action of the circle

$$T_q^1 = \{\cos t + q \sin t \,|\, t \in \mathbb{R}\} \subset SU(2) \tag{3.15}$$

on X is $\mathbf{J}_q$-holomorphic, and (c) $SU(2)$ acts isometrically and permutes the Kaehler structures X_q according to the standard action of $SU(2) = \widetilde{SO}(3)$ on S^2.

We can also rewrite the conditions (i)-(iii) at the infinitesimal level. Indeed the Lie algebra of the multiplicative group $\mathbb{H}^*$ is $\mathbb{H}$ with Lie bracket $[u, v] = uv - vu$. We have a standard basis $\mathbf{j}_0, \mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$ of $\mathbb{H}$ with

$$\mathbf{j}_0 = 1, \quad \mathbf{j}_1 = \mathbf{i}, \quad \mathbf{j}_2 = \mathbf{j}, \quad \mathbf{j}_3 = \mathbf{k}$$

and bracket relations

$$[\mathbf{j}_0, \mathbf{j}_a] = 0 \qquad \text{and} \qquad [\mathbf{j}_a, \mathbf{j}_b] = 2\varepsilon_{abc}\mathbf{j}_c \tag{3.16}$$

Differentiating the $\mathbb{H}^*$-action we get an infinitesimal vector field action

$$\psi : \mathbb{H} \to \mathfrak{Vect}\, X, \qquad q \mapsto \psi^q \tag{3.17}$$

We put

$$\eta = \psi^{-1} \qquad \text{and} \qquad \theta_a = \psi^{-\mathbf{j}_a}, \quad a = 1, 2, 3 \tag{3.18}$$

Then η is the infinitesimal generator of the $\mathbb{R}^+$-action and θ_a is the infinitesimal generator of the T_q^1-action. The bracket relations are

$$[\eta, \theta_a] = 0, \qquad [\theta_a, \theta_b] = -2\varepsilon_{abc}\theta_c \tag{3.19}$$

Now we can give an equivalent version of the conditions (3.13) on γ: ψ is $\mathbb{H}$-linear, i.e.,

$$\theta_a = \mathbf{J}_a\eta, \quad a = 1, 2, 3 \tag{3.20}$$

and also

$$\begin{aligned}
\mathcal{L}_\eta\omega_a &= k\omega_a, & \mathcal{L}_\eta\mathbf{J}_a &= 0, & \mathcal{L}_\eta g &= kg \\
\mathcal{L}_{\theta_a}\omega_b &= -2\tilde{\varepsilon}_{abc}\omega_c, & \mathcal{L}_{\theta_a}\mathbf{J}_b &= -2\tilde{\varepsilon}_{abc}\mathbf{J}_c, & \mathcal{L}_{\theta_a}g &= 0
\end{aligned} \tag{3.21}$$

In (3.21), we do not require a, b, c distinct, but instead we define

$$\tilde{\varepsilon}_{abc} = \begin{cases} sgn(abc) & \text{if } a, b, c \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

We call a vector field action (3.17) satisfying (3.18)-(3.21) *infinitesimally conical* with weight k.

Lemma 3.1 *Any infinitesimally conical vector field action of $\mathbb{H}$ on X necessarily has weight $k = 2$.*

Proof If a, b, c are distinct then (3.20) and (2.1) give

$$\theta_a \lrcorner \omega_b = -\varepsilon_{abc}\, \eta \lrcorner \omega_c \tag{3.22}$$

But then

$$-2\varepsilon_{abc}\omega_c = \mathcal{L}_{\theta_a}\omega_b = \mathrm{d}(\theta_a \lrcorner \omega_b) = -\varepsilon_{abc}\mathrm{d}(\eta \lrcorner \omega_c) = -\varepsilon_{abc}\mathcal{L}_\eta\omega_c = -k\varepsilon_{abc}\omega_c$$

Hence $k = 2$.   □

So any hyperkaehler cone necessarily has weight 2. From now on, we assume $k = 2$ in (3.13) and (3.21).

4   The Hyperkaehler Potential

In this section, we explain, on the global level, the relation between the hyperkaehler cone structure and the hyperkaehler potential. This was worked out locally in [Sw]; see also [HKLR, pg. 553] for part of this.

Corresponding to each complex structure $\mathbf{J}_q$ on X, we have the decomposition $\mathrm{d} = \partial_q + \overline{\partial}_q$ of the exterior derivative into $(1,0)$ and $(0,1)$ parts. We put

$$\mathrm{d}_q^c = -\frac{1}{2}\mathbf{J}_q\mathrm{d} = -\frac{i}{2}(\partial_q - \overline{\partial}_q) \tag{4.1}$$

A global *hyperkaehler potential* on X is a smooth function $\rho : X \to \mathbb{R}$ which is a simultaneous Kaehler potential for each Kaehler structure X_q, i.e.,

$$\omega_q = i\partial_q\overline{\partial}_q\rho = \mathrm{d}\mathrm{d}_q^c\rho \tag{4.2}$$

for all $q \in S^2$. It follows easily that ρ is a hyperkaehler potential iff ρ is a Kaehler potential for X_1, X_2 and X_3.

Proposition 4.1 *Suppose $(X, \mathbf{J}_q, \omega_q, g)$ admits a global hyperkaehler potential $\rho : X \to \mathbb{R}$. Let η be the vector field on X defined by*

$$\eta \lrcorner g = \mathrm{d}\rho \tag{4.3}$$

and set $\theta_q = \mathbf{J}_q\eta$.

Then ρ, after perhaps being modified by adding a constant, satisfies $\eta\rho = 2\rho$ so that ρ is homogeneous of weight 2. The vector fields $\eta, \theta_1, \theta_2, \theta_3$ define an infinitesimally conical vector field action of $\mathbb{H}$ on X where $\theta_1, \theta_2, \theta_3$ define the infinitesimal $\mathfrak{so}(3)$-action.

The potential ρ is $SO(3)$-invariant. For each $q \in S^2$, the Hamiltonian flow of ρ with respect to ω_q integrates the vector field θ_q. In other words, we have

$$\theta_q \lrcorner \omega_q + \mathrm{d}\rho = 0 \tag{4.4}$$

so that ρ is a simultaneous moment map for each infinitesimal S^1-action defined by θ_q.

Proof To begin with, we observe that if a vector field η and a function ρ satisfy (4.3) then

$$\rho \text{ is a hyperkaehler potential} \quad \Longleftrightarrow \quad \mathcal{L}_\eta\omega_a = 2\omega_a \text{ for } a = 1, 2, 3 \tag{4.5}$$

Indeed the computation

$$\langle \eta \lrcorner \omega_a, \xi \rangle = \omega_a(\eta, \xi) = g(\mathbf{J}_a\eta, \xi) = -g(\eta, \mathbf{J}_a\xi) = \langle -\mathrm{d}\rho, \mathbf{J}_a\xi \rangle = \langle -\mathbf{J}_a\mathrm{d}\rho, \xi \rangle = \langle 2\mathrm{d}_a^c\rho, \xi \rangle$$

gives $\eta \lrcorner \omega_a = 2\mathrm{d}_a^c\rho$. Applying d to this we get

$$\mathcal{L}_\eta\omega_a = 2\mathrm{d}\mathrm{d}_a^c\rho$$

This implies (4.5).

Now suppose ρ is a hyperkaehler potential. We need to prove all the relations in (3.21); in fact we can ignore the two involving the metric g as they are redundant. The relations $\mathcal{L}_\eta\omega_a = 2\omega_a$ are done in (4.5). These imply the relations $\mathcal{L}_\eta\mathbf{J}_b = 0$ because we can apply $\mathcal{L}_\eta$

to (2.1). This in turn gives $[\eta, \theta_a] = \mathcal{L}_\eta(\mathbf{J}_a \eta) = 0$. To prove the second line of relations in (3.21) we first observe that $g(u, v) = \omega_a(u, \mathbf{J}_a v) = \omega_a(-\mathbf{J}_a u, v)$ gives

$$-\theta_a \lrcorner \omega_a = \eta \lrcorner g \tag{4.6}$$

So

$$\mathcal{L}_{\theta_a} \omega_a = \mathrm{d}(\theta_a \lrcorner \omega_a) = -\mathrm{d}(\eta \lrcorner g) = \mathrm{d}^2 \rho = 0$$

For a, b, c distinct we find using (3.22)

$$\mathcal{L}_{\theta_a} \omega_b = \mathrm{d}(\theta_a \lrcorner \omega_b) = -\varepsilon_{abc} \mathrm{d}(\eta \lrcorner \omega_c) = -\varepsilon_{abc} \mathcal{L}_\eta \omega_c = -2\varepsilon_{abc} \omega_c$$

Next, applying $\mathcal{L}_{\theta_c}$ to (2.1) we find that $\mathcal{L}_{\theta_c} \mathbf{J}_a = -2\varepsilon_{abc} \mathbf{J}_b$. This proves the six independent relations in (3.21). Also

$$[\theta_a, \theta_b] = \mathcal{L}_{\theta_a} \theta_b = \mathcal{L}_{\theta_a}(\mathbf{J}_b \eta) = -2\varepsilon_{abc} \mathbf{J}_c \eta = -2\varepsilon_{abc} \theta_c$$

Thus we have an infinitesimally conical $\mathbb{H}$-action.

Now applying $\mathcal{L}_\eta$ to (4.3) we find $\mathcal{L}_\eta(\mathrm{d}\rho) = 2\mathrm{d}\rho$. So $\eta \mathrm{d}\rho = 2\mathrm{d}\rho$ and therefore $\eta\rho = 2\rho + C$ where C is constant. We can replace ρ by $\rho + C/2$ so that $\eta\rho = 2\rho$. Next applying $\mathcal{L}_{\theta_a}$ to (4.3) we find $\mathcal{L}_{\theta_a}(\mathrm{d}\rho) = 0$. So $\mathcal{L}_{\theta_a}\rho = C_a$ where C_a is constant. But then it follows from the semisimplicity of $\mathfrak{su}(2)$, in particular from the relations $[\theta_a, \theta_b] = -2\varepsilon_{abc}\theta_c$, that $C_1 = C_2 = C_3 = 0$. Hence $\theta_a \rho = 0$.

Finally, it suffices to check (4.4) for any single $q \in S^2$ because of the $\mathfrak{so}(3)$-action. Clearly $\theta_a \lrcorner \omega_a + \mathrm{d}\rho = 0$ follows by (4.3) and (4.6).   □

We may think of a hyperkaehler $\mathbb{H}^*$-conical structure on X as a family of Kaehler cone structures on X which satisfy additional properties. Indeed, for each $q \in S^2$, X_q is a Kaehler cone with respect to the action of $\mathbb{C}_q^* = \mathbb{R}^+ \times \mathcal{C}_q$ where the action of $\mathcal{C}_q$ integrates θ_q.

A converse to Proposition 4.1 follows easily from the proof.

Corollary 4.2 *Suppose $H^1_{deRham}(X) = 0$ and X admits an infinitesimally conical vector field action of $\mathbb{H}$. Then (4.3) has a smooth solution ρ on X (unique up to the addition of a constant function), and ρ is a global hyperkaehler potential. The further condition $\eta\rho = 2\rho$ uniquely determines ρ.*

Proof It is enough because of (4.5) and the relation $\mathcal{L}_\eta \omega_a = 2\omega_a$ in (3.21) to produce a solution ρ to (4.3). Since $H^1_{deRham}(X) = 0$, the problem reduces to showing that the 1-form $\eta \lrcorner g$ is closed. This is easy: the general fact (4.6) and one of the relations in (3.21) give $\mathrm{d}(\eta \lrcorner g) = -\mathcal{L}_{\theta_a} \omega_a = 0$.   □

Example 4.3 We continue discussing $X = \mathbb{H}^n$ from Examples 2.1 and 2.2. The left multiplication action of $\mathbb{H}^*$ on X given by

$$h \circ (q_1, \ldots, q_n) = (hq_1, \ldots, hq_n)$$

makes X into a hyperkaehler cone of weight 2. The natural $SO(3)$-action on the 2-sphere $\mathcal{S}_X$ of Kaehler structures is induced by the $SU(2)$-action on X defined by left multiplication by quaternions of unit norm. The infinitesimal generator η of the $\mathbb{R}^+$-action and the corresponding hyperkaehler potential ρ are

$$\eta = \sum_{r,s} x_s^r \frac{\partial}{\partial x_s^r} \qquad \text{and} \qquad \rho = \frac{1}{2}\sum_{r,s}(x_s^r)^2 = \frac{1}{2}\sum_s |q_s|^2 \tag{4.7}$$

The vector fields θ_a, $a = 1, 2, 3$, are

$$\theta_a = x_s^0 \frac{\partial}{\partial x_s^a} - x_s^a \frac{\partial}{\partial x_s^0} + x_s^b \frac{\partial}{\partial x_s^c} - x_s^c \frac{\partial}{\partial x_s^b} \tag{4.8}$$

where (abc) is cyclic.

Let $\mathbb{H}^{im} = \mathbf{i}\mathbb{R} \oplus \mathbf{j}\mathbb{R} \oplus \mathbf{k}\mathbb{R}$. We have a weight-2 left action of $\mathbb{H}^*$ on $\mathbb{H}^{im}$ defined by

$$h \bullet w = |h|^2(hwh^{-1}) \tag{4.9}$$

This is just the product of the degree 2 scaling action of $\mathbb{R}^+$ with the spin 1 action of $SU(2)$. We then get a tensor product action of $\mathbb{H}^* \times U$ on

$$\mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u} = \mathbb{H}^{im} \otimes \mathfrak{u} \tag{4.10}$$

given by, for $w \in \mathbb{H}^{im}$ and $u \in \mathfrak{u}$,

$$(h, a) \cdot (wu) = (h \bullet w)\mathrm{Ad}_a u \tag{4.11}$$

Lemma 4.4 *Suppose a hyperkaehler cone X has a hyperkaehler action of U which commutes with the $\mathbb{H}^*$-action. Then the triple moment map (2.11) is equivariant under $\mathbb{H}^* \times U$ with respect to the action (4.11).*

Proof This follows immediately by transforming the equation $\xi^u \lrcorner \omega_a + \mathrm{d}\zeta_a^u = 0$ under the $\mathbb{H}^*$-action. $\qquad\qquad\square$

Lemma 4.5 *Let Z be a complex manifold with $H^1_{DeRham}Z = 0$. Then*

(i) *A smooth function $\rho : Z \to \mathbb{R}$ is pluriharmonic (i.e., $\partial\overline{\partial}\rho = 0$) if and only if ρ is the real part of a holomorphic function on Z. Suppose further a compact group H acts holomorphically on Z and this action complexifies to a transitive action of the complexified group $H_{\mathbb{C}}$ on Z. Then*

(ii) *The only H-invariant pluriharmonic smooth functions $\rho : X \to \mathbb{R}$ are the constants.*

(iii) *If ω_Z is a Kaehler form on Z, then an H-invariant Kaehler potential on Z, if it exists, is unique up to addition of a constant.*

Proof (i) This follows by a standard argument (see e.g. [Kra]). Indeed the 1-form $i(\partial - \overline{\partial})(\rho)$ is closed and so exact since $H^1_{DeRham}Z = 0$. So we can solve $i(\partial - \overline{\partial})(\rho) = \mathrm{d}\phi$ globally for a smooth real-valued function ϕ. Then $i\partial\rho = \partial\phi$ and $i\overline{\partial}\rho = i\overline{\partial}\phi$. But then $f = \rho - i\phi$ satisfies $\overline{\partial}f = 0$ which means f is holomorphic. (ii) By (i), we have $\rho = \mathrm{Re}\,f$ for some holomorphic function f. We see easily that f is H-invariant. But then f is $H_{\mathbb{C}}$-invariant. Now the transitivity assumption forces f, and hence ρ, to be constant. (iii) follows from (ii) since the difference of any two potentials is pluriharmonic. $\square$

An immediate consequence is

Proposition 4.6 *Suppose X is as in Proposition* 4.1 *and X carries a hyperkaehler action of U which commutes with the infinitesimal $\mathbb{H}$-action. Then the homogeneous degree* 2 *solution ρ to* (4.3) *is, up to addition of a constant, the unique U-invariant hyperkaehler potential on X.*

5   Instantons and the Kronheimer Model of a Complex Nilpotent Orbit

In this section we recall some main results from [Kr]. First we construct the instanton space $\mathcal{M}(\kappa)$, and then we explain how $\mathcal{M}(\kappa)$ is isomorphic to a complex nilpotent orbit $\mathcal{O}$.

The subspace $\mathbb{H}^{im}$ of pure imaginary quaternions is a Lie subalgebra of $\mathbb{H}$ isomorphic to $\mathfrak{so}(3)$; cf. (3.16). From now on, we may identify $\mathfrak{so}(3)$ with $\mathbb{H}^{im}$. The adjoint action of $SO(3)$ identifies with the standard spin 1 action of $SO(3)$ on $\mathbb{H}^{im}$, which we write as $(\tau, w) \mapsto \tau * w$.

Let $L(\mathfrak{so}(3), \mathfrak{u})$ be the space of $\mathbb{R}$-linear maps

$$A : \mathfrak{so}(3) \to \mathfrak{u}, \qquad w \mapsto A_w \tag{5.1}$$

An element $A(t)$ of the path space $\mathcal{P} = C^\infty(\mathbb{R}, L(\mathfrak{so}(3), \mathfrak{u}))$ is given by a triple

$$A(t) = (A_1(t), A_2(t), A_3(t))$$

where $A_a(t) = A_{-\mathbf{j}_a}(t)$. The adjoint actions of $SO(3)$ and U define a representation of $SO(3) \times U$ on $L(\mathfrak{so}(3), \mathfrak{u})$. This gives a representation of $SO(3) \times U$ on $\mathcal{P}$ defined by

$$(\tau, u) \cdot A_w(t) = \mathrm{Ad}_u \cdot A_{\tau^{-1} * w}(t) \tag{5.2}$$

Let $\mathcal{M} \subset \mathcal{P}$ be the subspace of paths $A(t)$ which satisfy the system of three differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} A_a = -2A_a - [A_b, A_c] \tag{5.3}$$

where (abc) is a cyclic permutation of $1, 2, 3$. Now let

$$\kappa : \mathfrak{so}(3) \to \mathfrak{u} \tag{5.4}$$

be a non-zero Lie algebra homomorphism; then κ is 1-to-1. We put

$$e_a = \kappa(-\mathbf{j}_a), \qquad a = 1, 2, 3$$

Let $\mathcal{M}(\kappa) \subset \mathcal{M}$ be the subspace of paths satisfying the boundary conditions

$$\lim_{t \to \infty} A(t) = 0, \qquad \lim_{t \to -\infty} A(t) \in C(\kappa) \tag{5.5}$$

where $C(\kappa) \subset L(\mathfrak{so}(3), \mathfrak{u})$ is the space of Lie algebra homomorphisms which are U-conjugate to κ. The action of $SO(3) \times U$ on $\mathcal{P}$ leaves stable $\mathcal{M}$ and $\mathcal{M}(\kappa)$. Thus we have a smooth action

$$SO(3) \times U \to \mathrm{Diff}\,\mathcal{M}(\kappa) \tag{5.6}$$

An *instanton* is an element of $\mathcal{M}(\kappa)$. We often write an instanton $A(t)$ simply as "A" with the time dependence being understood.

It follows from the equations (5.3) (see [Kr]) that we have three well-defined U-equivariant mappings

$$\zeta_1, \zeta_2, \zeta_3 : \mathcal{M}(\kappa) \to \mathfrak{u} \tag{5.7}$$

given by

$$\zeta_a(A) = \frac{1}{2} \lim_{t \to \infty} e^{2t} A_a(t) \tag{5.8}$$

We can arrange these three maps $\zeta_1, \zeta_2, \zeta_3$ into the single map

$$\zeta = (\mathbf{i}\zeta_1, \mathbf{j}\zeta_2, \mathbf{k}\zeta_3) : \mathcal{M}(\kappa) \to \mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u} \tag{5.9}$$

We have a natural (tensor product) action of $SO(3) \times U$ on $\mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u}$ because of (4.10). We see easily that ζ is $(SO(3) \times U)$-equivariant.

Each triple $d = (d_1, d_2, d_3)$ lying in $C(\kappa)$ gives rise to the "model" instanton

$$D(t) = (1 + e^{2t})^{-1}(d_1, d_2, d_3)$$

Then $\zeta_a(D(t)) = \frac{1}{2}d_a$.

It is easy to check that we have an action of $\mathbb{R}^+$ on $\mathcal{M}(\kappa)$ given by

$$\lambda \diamond A(t) = A\left(t - \frac{1}{2}\log \lambda\right) \tag{5.10}$$

This commutes with the action of $SO(3) \times U$.

The complexification of κ is a complex Lie algebra embedding

$$\kappa_{\mathbb{C}} : \mathfrak{so}(3, \mathbb{C}) \to \mathfrak{g}$$

The set $\mathcal{N}$ of all nilpotent elements in $\mathfrak{so}(3, \mathbb{C})$ is

$$\mathcal{N} = \{a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \,|\, a^2 + b^2 + c^2 = 0\}$$

and $\mathcal{N} - \{0\}$ is a single orbit under the adjoint action of $SO(3, \mathbb{C})$. It follows that all the elements $\kappa(z)$, $z \in \mathcal{N} - \{0\}$, lie in a single adjoint orbit $\mathcal{O}$ in $\mathfrak{g}$. Then $\mathcal{O}$ consists of nilpotent elements in $\mathfrak{g}$; i.e., $\mathcal{O}$ is a complex nilpotent orbit.

In particular then

$$\mathcal{O} = G \cdot (e_2 + ie_3) = G \cdot (e_3 + ie_1) = G \cdot (e_1 + ie_2) \tag{5.11}$$

Then $\mathcal{O}$ inherits a complex structure $\mathbf{I}$ from the natural embedding of $\mathcal{O}$ into $\mathfrak{g}$; we call this embedding

$$\Phi_{\mathbf{I}} : \mathcal{O} \to \mathfrak{g} \tag{5.12}$$

Or, equivalently, $\mathbf{I}$ is induced by the G-action. $\mathbf{I}$ and the G-invariant KKS holomorphic symplectic form Σ make $\mathcal{O}$ into a holomorphic symplectic manifold

$$(\mathcal{O}, \mathbf{I}, \Sigma) \tag{5.13}$$

We recall that Σ is the unique holomorphic symplectic form on $\mathcal{O}$ such that the adjoint action of G on $\mathcal{O}$ is holomorphic Hamiltonian with moment map $\Phi_{\mathbf{I}}$. In terms of the holomorphic component function $\Phi_{\mathbf{I}}^z$, $z \in \mathfrak{g}$, defined by $\Phi_{\mathbf{I}}^z(w) = (z, w)_{\mathfrak{g}}$, this means that the Σ-Hamiltonian flow of the functions $\Phi_{\mathbf{I}}^z$ gives the G-action and the map

$$\mathfrak{g} \to R_{\mathbf{I}-hol}(\mathcal{O}), \qquad z \mapsto \Phi_{\mathbf{I}}^z \tag{5.14}$$

is a complex Lie algebra homomorphism with respect to the Poisson bracket on $R_{\mathbf{I}-hol}(\mathcal{O})$ defined by Σ.

The space $\mathcal{M}(\kappa)$ has a natural U-invariant hyperkaehler structure $(g, \mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3, \omega_1, \omega_2, \omega_3)$; see [Kr, Remark 2, pg 476] and [H]. Kronheimer discovered

Theorem 5.1 *[Kr]*

(i) *The map ζ in (5.9) is an $(SO(3) \times U)$-equivariant smooth embedding of manifolds.*

(ii) *The three maps $\zeta_1, \zeta_2, \zeta_3$ are the moment maps for the U-action with respect to the three Kaehler forms $\omega_1, \omega_2, \omega_3$ on $\mathcal{M}(\kappa)$.*

(iii) *For $a = 1, 2, 3$, the holomorphic moment map $\Phi_a : \mathcal{M}(\kappa) \to \mathfrak{g}$ given by (2.17) is 1-to-1 and has image equal to $\mathcal{O}$. Thus we get a G-equivariant holomorphic symplectic isomorphism*

$$\Phi_a : \mathcal{M}(\kappa) \to \mathcal{O} \tag{5.15}$$

from $(\mathcal{M}(\kappa), \mathbf{J}_a, \Omega_a)$ to $(\mathcal{O}, \mathbf{I}, \Sigma)$. Here G acts on $\mathcal{M}(\kappa)$ by the $\mathbf{J}_a$-complexification of the U-action.

(iv) *The $SO(3)$-action*

$$SO(3) \to \mathrm{Diff}\,\mathcal{M}(\kappa) \tag{5.16}$$

preserves the Riemannian metric g and induces the standard transitive action of $SO(3)$ on the 2-sphere $\mathcal{S}_{\mathcal{M}(\kappa)}$ of Kaehler structures on $\mathcal{M}(\kappa)$.

In fact, (ii) determines uniquely the U-invariant hyperkaehler structure on $\mathcal{M}(\kappa)$. Let $\mathcal{C}_a = \mathcal{C}_{\mathbf{j}_a}$. We further note

Corollary 5.2

(i) *The map ζ is $\mathbb{R}^+$-equivariant with respect to* (5.10) *and the Euler scaling action on* $\mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u}$, *i.e.,*

$$\zeta(\lambda \diamond A) = \lambda \zeta(A) \tag{5.17}$$

(ii) *The isomorphism Φ_a intertwines the product action of $\mathbb{R}^+ \times \mathcal{C}_a$ on $\mathcal{M}(\kappa)$ with the Euler scaling action of $\mathbb{C}^*$ on $\mathcal{O}$.*

(iii) *The $SO(3)$-action on $\mathcal{M}(\kappa)$ is free.*

Proof (i) and (ii) are routine to verify. The Euler $\mathbb{C}^*$-action on $\mathcal{O}$ has only trivial isotropy groups, and so (ii) implies that the action of $\mathcal{C}_a$, and hence of all of $SO(3)$, has only trivial isotropy groups. A compact group action with only trivial isotropy groups is necessarily free. □

Next we define a left action of $\mathbb{H}^*$ on $\mathcal{M}(\kappa)$ by

$$h \cdot A_w = |h|^2 \diamond A_{h^{-1}wh} \tag{5.18}$$

This is the product of the square of the $\mathbb{R}^+$-action (5.10) with the $SU(2)$-action defined by the $SO(3)$-action on $\mathcal{M}(\kappa)$ the spin homomorphism

$$SU(2) \to SO(3) \tag{5.19}$$

We also have an $\mathbb{H}^*$-action on $\mathbf{i}\mathfrak{u} \oplus \mathbf{j}\mathfrak{u} \oplus \mathbf{k}\mathfrak{u}$, $(h, p) \mapsto h \bullet p$ defined by (4.9).

Let η be the infinitesimal generator of the square of the action (5.10).

Corollary 5.3 *The left $\mathbb{H}^*$-action on $\mathcal{M}(\kappa)$ defined in* (5.18) *satisfies*

$$\zeta(h \cdot A) = h \bullet \zeta(A) \tag{5.20}$$

commutes with the action of U, and gives $\mathcal{M}(\kappa)$ the structure of a hyperkaehler cone.

Proof The $\mathbb{H}^*$-action on $\mathcal{M}(\kappa)$ clearly commutes with the U-action and satisfies (5.8) because of the $SO(3)$-equivariance in Theorem 5.1(i) and the $\mathbb{R}^+$-equivariance in Corollary 5.2(i).

Next we check the hyperkaehler cone axioms (3.13). By Theorem 5.1, the $SU(2)$-action transforms the tensors $\mathbf{J}_q$, ω_q, g according to (3.13). The action of the subgroup $\mathbb{C}_a^* = \mathbb{C}_{\mathbf{j}_a}^*$ of $\mathbb{H}^*$ identifies (because of Corollary 5.2(ii)) with the *square* of the Euler $\mathbb{C}^*$-action on $\mathcal{O}$ under Φ_a. Hence the action of $\mathbb{C}_a^*$ is $\mathbf{J}_a$-holomorphic. Furthermore, this implies that the action of the $\mathbb{R}^+$ subgroup of $\mathbb{H}^*$ preserves $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$, and transforms $\Omega_1, \Omega_2, \Omega_3$, and hence $\omega_1, \omega_2, \omega_3$, by the degree 2 character. Thus the $\mathbb{R}^+$-action also satisfies the axioms (3.13). □

Let η be the infinitesimal generator of the action on $\mathcal{M}(\kappa)$ of the $\mathbb{R}^+$-subgroup of $\mathbb{H}^*$, i.e., of the square of the action (5.10). We will say a function f on $\mathcal{M}(\kappa)$ is *homogeneous of degree r* if this $\mathbb{R}^+$-action transforms f by the degree r character, i.e., $\eta f = rf$.

Corollary 5.4 *The hyperkaehler manifold $\mathcal{M}(\kappa)$ admits a U-invariant hyperkaehler potential ρ, unique up to addition of a constant. The further condition that ρ is homogeneous of degree 2 determines ρ uniquely.*

Proof This follows by Corollary 5.3 and Proposition 4.6 as soon as we check that

$$H^1_{DeRham}\mathcal{M}(\kappa) = 0$$

But $\mathcal{M}(\kappa)$ is diffeomorphic to $\mathcal{O}$ by Theorem 5.1(iii) and we have the well-known fact

Lemma 5.5 *The fundamental group of a complex nilpotent orbit $\mathcal{O}$ is finite. Consequently $H^1_{DeRham}\mathcal{O} = 0$.*

This finishes the proof of Corollary 5.4. $\qquad\square$

6   Complex Conjugation on the Instanton Space $\mathcal{M}(\kappa)$

We will introduce a family of complex conjugation maps on $\mathcal{M}(\kappa)$.

We start with the non-trivial Lie algebra involution ς of $\mathfrak{so}(3)$ given by

$$\varsigma(a\mathbf{j}_1 + b\mathbf{j}_2 + c\mathbf{j}_3) = -a\mathbf{j}_1 + b\mathbf{j}_2 - c\mathbf{j}_3 \tag{6.1}$$

(The exact choice is immaterial, as any two choices are $SO(3)$-conjugate.) We assume now that the induced involution on $\kappa(\mathfrak{so}(3))$ extends to a Lie algebra involution ϑ of $\mathfrak{u}$. Then ϑ determines a splitting

$$\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{b} \tag{6.2}$$

where $\mathfrak{a}$ is the 1-eigenspace of ϑ and $\mathfrak{b}$ is the (-1)-eigenspace of ϑ. This gives the bracket relations

$$[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}, \qquad [\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}, \qquad [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a} \tag{6.3}$$

Lemma 6.1 *The involution ϑ of $\mathfrak{u}$ induces a hyperkaehler involution Θ of $\mathcal{M}(\kappa)$ which commutes with the $SO(3)$-action.*

Proof It follows from the instanton differential equations (5.3) and the boundary conditions (5.5) that any Lie algebra automorphism of $\mathfrak{u}$ induces a hyperkaehler diffeomorphism of $\mathcal{M}(\kappa)$. Clearly Θ commutes with the $SO(3)$-action. $\qquad\square$

Next we extend ϑ in a $\mathbb{C}$-antilinear manner to get a real Lie algebra involution ν of $\mathfrak{g}$. We have

$$\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u} = \mathfrak{a} \oplus \mathfrak{b} \oplus i\mathfrak{a} \oplus i\mathfrak{b} \tag{6.4}$$

and so for $x, x' \in \mathfrak{a}$ and $y, y' \in \mathfrak{b}$ we get

$$\nu(x + y + ix' + iy') = x - y - ix' + iy' \tag{6.5}$$

The point is that ν is a complex conjugation map on the Lie algebra $\mathfrak{g}$. The corresponding real form of $\mathfrak{g}$ consisting of ν-fixed vectors is

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{a} \oplus i\mathfrak{b} \tag{6.6}$$

Now $\mathfrak{g}_{\mathbb{R}}$ is a non-compact real form of $\mathfrak{g}$, unlike $\mathfrak{u}$ which is a compact real form. Indeed the Killing form of $\mathfrak{g}$, being negative-definite on $\mathfrak{u}$, is clearly indefinite on $\mathfrak{g}_{\mathbb{R}}$. From now on, we call ν "complex conjugation" and we put $\overline{z} = \nu(z)$ for $z \in \mathfrak{g}$.

Clearly $\mathfrak{g}_{\mathbb{R}}$ is compatible with κ in the sense that $\kappa(\mathfrak{so}(3))$ is complex conjugation stable in $\mathfrak{u}$; in fact $\kappa(\varsigma(w)) = \vartheta(\kappa(w))$ for $w \in \mathfrak{so}(3)$. Hence complex conjugation preserves $\mathcal{O}$ inside $\mathfrak{g}$. We conclude

Lemma 6.2 *Our map $\nu : \mathfrak{g} \to \mathfrak{g}$ induces an antiholomorphic diffeomorphism of $\mathcal{O}$, which we again call ν. So $\nu : \mathcal{O} \to \mathcal{O}$ is a complex conjugation map.*

See §7 and (7.7) for how this arises in practice. We work with (6.6) now because the essential symmetry of the picture is more apparent this way (just as for the Sekiguchi correspondence).

For $a = 1, 2, 3$, we call pull back complex conjugation on $\mathcal{O}$ through the holomorphic isomorphism Φ_a in (5.15) to get an involution

$$\gamma_a : \mathcal{M}(\kappa) \to \mathcal{M}(\kappa), \qquad \gamma_a = \Phi_a^* \nu \tag{6.7}$$

So $\Phi_a(\gamma_a A) = \nu(\Phi_a A)$. Then γ_a is $\mathbf{J}_a$-antiholomorphic and so γ_a defines a complex conjugation map on $\mathcal{M}(\kappa)$ with respect to $\mathbf{J}_a$.

Our aim is to figure out how the hyperkaehler data transforms under the involutions γ_a. To this end, we find a formula for γ_a in terms of Θ and the $SO(3)$-action on $\mathcal{M}(\kappa)$. In particular, the rotation in $\mathbb{H}^{im}$ about the $\mathbf{j}_a$-axis defines via (5.16) an isometric involution

$$R_a : \mathcal{M}(\kappa) \to \mathcal{M}(\kappa)$$

Proposition 6.3 *Let (abc) be a cyclic permutation of $1, 2, 3$. Then*

$$\gamma_a = \Theta R_b \tag{6.8}$$

Proof Let $\vartheta : \mathfrak{g} \to \mathfrak{g}$ be the involution defined by $\mathbb{C}$-linearly extending $\vartheta : \mathfrak{u} \to \mathfrak{u}$. Then ϑ induces a biholomorphic diffeomorphism of $\mathcal{O}$, which we again call ϑ. It is easy to see that

$$\Theta = \Phi_a^* \vartheta \tag{6.9}$$

The composition $\beta = \nu\vartheta$ is the $\mathbb{C}$-antilinear involution $\beta : \mathfrak{g} \to \mathfrak{g}$ with fixed space $\mathfrak{u}$, i.e, β is complex conjugation with respect to $\mathfrak{u}$. So β induces an antiholomorphic involution β of $\mathcal{O}$. We now see that (6.8) holds if and only if

$$R_b = \Phi_a^* \beta \tag{6.10}$$

To compute $\Phi_a^* \beta$, we first write out the map Φ_a according to (2.16):

$$\Phi_a = \zeta_b + i\zeta_c \tag{6.11}$$

It follows that the diffeomorphism $\tilde{\beta} = \Phi_a^* \beta$ of the instanton space $\mathcal{M}(\kappa)$ satisfies

$$\tilde{\beta}^* \zeta_b = \zeta_b \qquad \text{and} \qquad \tilde{\beta}^* \zeta_c = -\zeta_c$$

But the rotation R_b transforms ζ_b and ζ_c in exactly the same way. This proves (6.10). □

Our formula (6.8) immediately implies, because of Theorem 5.1(iv):

Corollary 6.4 *Let (abc) be a cyclic permutation of $1, 2, 3$. Then the involution γ_a is an isometry of $\mathcal{M}(\kappa)$ which preserves $(\mathbf{J}_b, \omega_b)$ but negates $(\mathbf{J}_a, \omega_a)$ and $(\mathbf{J}_c, \omega_c)$. Furthermore, γ_a commutes with the action of $\mathcal{C}_b$.*

In particular γ_a is a complex conjugation map with respect to $\mathbf{J}_a$ which is also g-isometric, $\mathbf{J}_b$-holomorphic, ω_b-symplectic and $\mathcal{C}_b$-invariant.

Let $\mathcal{M}(\kappa)^{\gamma_a}$ be the subset of $\mathcal{M}(\kappa)$ fixed by γ_a.

Corollary 6.5

(i) $\mathcal{M}(\kappa)^{\gamma_1}$ *is a smooth submanifold of $\mathcal{M}(\kappa)$ with finitely many connected components, and each component has half the dimension of $\mathcal{M}(\kappa)$.*

(ii) $\mathcal{M}(\kappa)^{\gamma_1}$ *is a symplectic real form of $\mathcal{M}(\kappa)$ with respect to $(\mathbf{J}_1, \Omega_1)$.*

(iii) $\mathcal{M}(\kappa)^{\gamma_1}$ *is a Kaehler submanifold of $\mathcal{M}(\kappa)$ with respect to $(g, \mathbf{J}_2, \omega_2)$.*

(iv) $\mathcal{M}(\kappa)^{\gamma_1}$ *is a complex Lagrangian submanifold of $\mathcal{M}(\kappa)$ with respect to $(\mathbf{J}_2, \Omega_2)$.*

Proof If $\alpha : M \to M$ is a finite-order automorphism of a smooth manifold, then the fixed space M^α is a smooth submanifold. The tangent space of M^α at m is the subspace of α-fixed vectors in the tangent space of M at m.

Now Corollary 6.4 implies that $\mathcal{M}(\kappa)^{\gamma_1}$ is a Kaehler submanifold with respect to $(\mathbf{J}_2, \omega_2)$. So $\mathcal{M}(\kappa)^{\gamma_1}$ is a $\mathbf{J}_2$-complex submanifold and the restriction of ω_2 to it is a real symplectic form π. Furthermore $\mathcal{M}(\kappa)^{\gamma_1}$ is a real form with respect to $\mathbf{J}_1$ and $\mathbf{J}_3$ and ω_1 and ω_3 vanish on $\mathcal{M}(\kappa)^{\gamma_1}$. So $\mathcal{M}(\kappa)^{\gamma_1}$ is Lagrangian with respect to $\Omega_2 = \omega_3 + i\omega_1$ and Ω_1 restricts to π. This proves (ii)-(iv).

Finally we complete the proof of (i). First, $\mathcal{M}(\kappa)^{\gamma_1}$, being a real form of $\mathcal{M}(\kappa)$, has half the dimension. Next, $\mathcal{M}(\kappa)^{\gamma_1}$ has only finitely many components since $\gamma_1 = \Phi_1^* \nu$ by (6.4.1), and so

$$\Phi_1(\mathcal{M}(\kappa)^{\gamma_1}) = \mathcal{O} \cap \mathfrak{g}_\mathbb{R} \tag{6.12}$$

But $\mathcal{O} \cap \mathfrak{g}_\mathbb{R}$ is a real algebraic variety and so has only finitely many components. $\square$

Corollary 6.5 above and the next result below encode Vergne's result in "hyperkaehler language".

Proposition 6.6 *We have*

$$\begin{aligned}
\Phi_1(\mathcal{M}(\kappa)^{\gamma_1}) &= \mathcal{O} \cap (\mathfrak{a} \oplus i\mathfrak{b}) \\
\Phi_2(\mathcal{M}(\kappa)^{\gamma_1}) &= \mathcal{O} \cap (\mathfrak{b} \oplus i\mathfrak{b}) \\
\Phi_3(\mathcal{M}(\kappa)^{\gamma_1}) &= \mathcal{O} \cap (\mathfrak{b} \oplus i\mathfrak{a})
\end{aligned} \tag{6.13}$$

We already observed in (6.12) that the first equality is clear, but the others reveal a subtle structure.

Proof We start by computing how the functions ζ_a^u, $u \in \mathfrak{u}$, transform under $\gamma = \gamma_1$. For any function f on $\mathcal{M}(\kappa)$, we put $f^\gamma = \gamma^* f = f \circ \gamma$.

By (6.7), we have $\gamma = \gamma_1 = \Theta R_b$. So $\gamma^* = R_b^* \Theta^*$. We find $\Theta^* \zeta_a^u = \zeta_a^{\vartheta u}$ and $R_b^* \zeta_a^u = t_a \zeta_a^u$ where $t_1 = t_3 = -1$ and $t_2 = 1$. Therefore, writing $u = x + y$ where $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, we get

$$\left(\zeta_1^{x+y} \right)^\gamma = \zeta_1^{-x+y}, \qquad \left(\zeta_2^{x+y} \right)^\gamma = \zeta_2^{x-y}, \qquad \left(\zeta_3^{x+y} \right)^\gamma = \zeta_3^{-x+y} \tag{6.14}$$

Let $u' = -x + y$ and $u'' = x - y$. Then (6.14) says

$$\zeta_1(\gamma A) = \zeta_1(A)', \qquad \zeta_2(\gamma A) = \zeta_2(A)'', \qquad \zeta_3(\gamma A) = \zeta_3(A)', \tag{6.15}$$

Now we can compute the involutions α_b, $b = 1, 2, 3$, of $\mathcal{O}$ defined by $\Phi_a^* \alpha_a = \gamma_1$. Of course $\alpha_1 = \nu$ by the definition (6.7) of γ_1. Using (2.16) and (6.15) we find

$$\begin{aligned}
\Phi_2(A) &= \zeta_3(A) + i\zeta_1(A) \Rightarrow \Phi_2(\gamma A) = \zeta_3(A)' + i\zeta_1(A)' \\
\Phi_3(A) &= \zeta_1(A) + i\zeta_2(A) \Rightarrow \Phi_3(\gamma A) = \zeta_1(A)' + i\zeta_2(A)''
\end{aligned} \tag{6.16}$$

Consequently, the involutions α_2, α_3 of $\mathcal{O}$ are induced by the involutions α_2, α_3 of $\mathfrak{g}$ given by, for $x_1, x_2 \in \mathfrak{a}$ and $y_1, y_2 \in \mathfrak{b}$,

$$\begin{aligned}
\alpha_2(x_1 + y_1 + ix_2 + iy_2) &= -x_1 + y_1 - ix_2 + iy_2 \\
\alpha_3(x_1 + y_1 + ix_2 + iy_2) &= -x_1 + y_1 + ix_2 - iy_2
\end{aligned} \tag{6.17}$$

So the fixed spaces are

$$\mathfrak{g}^{\alpha_1} = \mathfrak{a} \oplus i\mathfrak{b}, \qquad \mathfrak{g}^{\alpha_2} = \mathfrak{b} \oplus i\mathfrak{b}, \qquad \mathfrak{g}^{\alpha_3} = \mathfrak{b} \oplus i\mathfrak{a} \tag{6.18}$$

For $b = 1, 2, 3$, we have

$$\Phi_b(\mathcal{M}(\kappa)^\gamma) = \mathcal{O}^{\alpha_b} = \mathcal{O} \cap \mathfrak{g}^{\alpha_b} \tag{6.19}$$

This completes the proof. $\qquad \square$

We set

$$P^1 = \mathcal{O} \cap (\mathfrak{a} \oplus i\mathfrak{b}), \qquad P^2 = \mathcal{O} \cap (\mathfrak{b} \oplus i\mathfrak{b}), \qquad P^3 = \mathcal{O} \cap (\mathfrak{b} \oplus i\mathfrak{a}) \tag{6.20}$$

Now combining Proposition 6.6 with Corollary 6.5, we get

Corollary 6.7 *For $a = 1, 2, 3$, P^a is the disjoint union of finitely many connected components. Each component of P^a is a smooth real submanifold of $\mathcal{O}$ of half the dimension. Each component is stable under the dilation action of $\mathbb{R}^+$ on $\mathfrak{g}$.*

P^1 is a symplectic real form of $(\mathcal{O}, \mathbf{I}, \Sigma)$ and P^2 is a holomorphic Lagrangian submanifold of $(\mathcal{O}, \mathbf{I}, \Sigma)$.

Let U^ϑ be the subgroup of U commuting with ϑ; then U^ϑ is the subgroup with Lie algebra equal to $\mathfrak{a}$.

Let $\mathcal{R} \in SO(3)$ be the element which gives the cyclic permutation

$$\mathcal{R}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = c\mathbf{i} + a\mathbf{j} + b\mathbf{k} \tag{6.21}$$

Then $\mathcal{R}$ defines an automorphism $\mathcal{R} : \mathcal{M}(\kappa) \to \mathcal{M}(\kappa)$ by (5.16).

Corollary 6.8 *[Ve] We have $(\mathbb{R}^+ \times U^\vartheta)$-equivariant diffeomorphisms*

$$\Phi_b \Phi_a^{-1} : P^a \to P^b \tag{6.22}$$

This sets up a bijection between the connected components of P^a and those of P^b.

We get the following commutative diagram of diffeomorphisms where (abc) is a cyclic permutation of $1, 2, 3$:

$$
\begin{array}{ccc}
\mathcal{M}(\kappa)^{\gamma_a} & \stackrel{\mathcal{R}^{-1}}{\longrightarrow} & \mathcal{M}(\kappa)^{\gamma_b} \\
\downarrow{\scriptstyle \Phi_a} & & \downarrow{\scriptstyle \Phi_a} \\
P^a & \stackrel{\Phi_b \Phi_a^{-1}}{\longrightarrow} & P^b
\end{array}
\tag{6.23}
$$

Proof We have

$$
\mathcal{R} \cdot (A_1(t), A_2(t), A_3(t)) = (A_3(t), A_1(t), A_2(t))
\tag{6.24}
$$

and so we get the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(\kappa) & \stackrel{\mathcal{R}^{-1}}{\longrightarrow} & \mathcal{M}(\kappa) \\
\downarrow{\scriptstyle \Phi_a} & & \downarrow{\scriptstyle \Phi_a} \\
\mathcal{O} & \stackrel{\Phi_b \Phi_a^{-1}}{\longrightarrow} & \mathcal{O}
\end{array}
\tag{6.25}
$$

Now (6.23) follows by restriction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7 Real Nilpotent Orbits and the Vergne Diffeomorphism

From now on, we single out in Theorem 5.1(iii) the holomorphic symplectic isomorphism

$$
\Phi_1 : \mathcal{M}(\kappa) \to \mathcal{O}
\tag{7.1}
$$

Now by means of Φ_1 we transfer the hyperkaehler structure on $\mathcal{M}(\kappa)$ over to $\mathcal{O}$. Thus we get a set

$$
(g, \mathbf{I}, \mathbf{J}, \mathbf{K}, \omega_{\mathbf{I}}, \omega_{\mathbf{J}}, \omega_{\mathbf{K}})
\tag{7.2}
$$

of hyperkaehler data on $\mathcal{O}$ with a corresponding 2-sphere $\mathcal{S}$ of complex structures on $\mathcal{O}$. We call this the *instanton hyperkaehler structure* on $\mathcal{O}$.

By Theorem 5.1, $\mathbf{I} = (\Phi_1)_* \mathbf{J}_1$ is the natural complex structure on $\mathcal{O}$ discussed in §5 and the holomorphic symplectic KKS form is

$$
\Sigma = \omega_{\mathbf{J}} + i\omega_{\mathbf{K}}
\tag{7.3}
$$

The three moment maps $\zeta_1, \zeta_2, \zeta_3 : \mathcal{M}(\kappa) \to \mathfrak{u}$ transfer by means of Φ_1 to three U-equivariant moment maps

$$
\zeta_{\mathbf{I}} : \mathcal{O} \to \mathfrak{u}, \qquad \zeta_{\mathbf{J}} : \mathcal{O} \to \mathfrak{u}, \qquad \zeta_{\mathbf{K}} : \mathcal{O} \to \mathfrak{u}
\tag{7.4}
$$

We recover the natural embedding $\Psi : \mathcal{O} \to \mathfrak{g}$ from two of these maps as

$$
\Psi = \zeta_{\mathbf{J}} + i\zeta_{\mathbf{K}}
\tag{7.5}
$$

However the third moment map $\zeta_{\mathbf{I}}$ is a mystery.

The $SO(3)$-action (5.16) on $\mathcal{M}(\kappa)$ transfers by means of Φ_1 to a smooth action on $\mathcal{O}$

$$SO(3) \to \text{Diff}\,\mathcal{O} \tag{7.6}$$

which commutes with the U-action. We call this the *Kronheimer $SO(3)$-action* on $\mathcal{O}$.

We emphasize that this $SO(3)$-action depends on a choice of maximal compact subgroup of U of G. The Kronheimer $SO(3)$-action remains quite mysterious.

We can rewrite (6.6) as

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} \tag{7.7}$$

where $\mathfrak{k}_{\mathbb{R}} = \mathfrak{a}$ and $\mathfrak{p}_{\mathbb{R}} = i\mathfrak{b}$. This is a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g}_{\mathbb{R}}$ in the usual sense: $\mathfrak{k}_{\mathbb{R}}$ is a maximal compact Lie subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ is an ad $\mathfrak{k}_{\mathbb{R}}$-stable complementary subspace. Then (6.2) becomes

$$\mathfrak{u} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}} \tag{7.8}$$

In practice, one may start from (7.3.1) and then constructs $\mathfrak{u}$ by (7.8). Then $\mathfrak{k}_{\mathbb{R}}$ and $\mathfrak{p}_{\mathbb{R}}$ are real forms of $\mathfrak{k} = \mathfrak{a} \oplus i\mathfrak{a}$ and $\mathfrak{p} = \mathfrak{b} \oplus i\mathfrak{b}$ and the complexification of (7.7) is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{7.9}$$

Let $(\,,\,)_{\mathfrak{g}_{\mathbb{R}}}$ be the Killing form of $\mathfrak{g}_{\mathbb{R}}$; this coincides with the restriction to $\mathfrak{g}_{\mathbb{R}}$ of the complex Killing form $(\,,\,)_{\mathfrak{g}}$.

We know that Φ_1 identifies $\mathcal{M}(\kappa)^{\gamma_1}$ with $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ by (6.12) or Proposition 6.6. Let $\mathcal{O}_{\mathbb{R}}$ be a connected component of $\mathcal{O}$. Then $\mathcal{O}_{\mathbb{R}}$ is stable under the dilation action of $\mathbb{R}^+$ on $\mathfrak{g}_{\mathbb{R}}$. Moreover $\mathcal{O}_{\mathbb{R}}$ is an orbit under the adjoint action of $G_{\mathbb{R}}$. This follows easily since $\mathcal{O}_{\mathbb{R}}$ is a connected real form of $\mathcal{O}$ and $\mathcal{O}$ is an adjoint orbit of G.

Hence $\mathcal{O}_{\mathbb{R}}$ is a real nilpotent orbit of $G_{\mathbb{R}}$. Every real nilpotent orbit arises in this way. Let σ be the real $G_{\mathbb{R}}$-invariant KKS symplectic form on $\mathcal{O}_{\mathbb{R}}$. Let

$$\phi : \mathcal{O}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}} \tag{7.10}$$

be the natural embedding. Then σ is the unique symplectic form on $\mathcal{O}_{\mathbb{R}}$ such that the adjoint action of $G_{\mathbb{R}}$ on $\mathcal{O}_{\mathbb{R}}$ is Hamiltonian with moment map ϕ. In terms of the component function ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, defined by $\phi^w(z) = (w, z)_{\mathfrak{g}_{\mathbb{R}}}$, this means that the σ-Hamiltonian flow of the functions ϕ^w gives the $G_{\mathbb{R}}$-action and the map

$$\mathfrak{g}_{\mathbb{R}} \to C^\infty(\mathcal{O}_{\mathbb{R}}), \qquad w \mapsto \phi^w \tag{7.11}$$

is a Lie algebra homomorphism with respect to the Poisson bracket on $C^\infty(\mathcal{O}_{\mathbb{R}})$ defined by σ. We easily see that

Lemma 7.1 *The real nilpotent orbit $(\mathcal{O}_{\mathbb{R}}, \sigma)$ is a symplectic real form of the complex nilpotent orbit $(\mathcal{O}, \mathbf{I}, \Sigma)$. I.e., $\mathcal{O}_{\mathbb{R}}$ is a real form of $\mathcal{O}$ and the real part of Σ restricts to σ on $\mathcal{O}_{\mathbb{R}}$ while the imaginary part of Σ vanishes on $\mathcal{O}_{\mathbb{R}}$. Thus*

$$\sigma = \Sigma|_{\mathcal{O}_{\mathbb{R}}} \tag{7.12}$$

Theorem 7.2 $\mathcal{O}_{\mathbb{R}}$ *is a complex submanifold of* $\mathcal{O}$ *with respect to the complex structure* $\mathbf{J}$ *on* $\mathcal{O}$. *Consequently the Kaehler structure* $(g, \mathbf{J}, \omega_{\mathbf{J}})$ *on* $\mathcal{O}$ *defines by restriction a* $K_{\mathbb{R}}$-*invariant Kaehler structure on* $\mathcal{O}_{\mathbb{R}}$.

The Kaehler form $\omega_{\mathbf{J}}|_{\mathcal{O}_{\mathbb{R}}}$ *coincides with the KKS symplectic form* σ.

We call this Kaehler structure on $\mathcal{O}_{\mathbb{R}}$ the *instanton Kaehler structure*.

Proof By (7.12) and (7.3), we have

$$\sigma = \mathrm{Re}\,\Sigma|_{\mathcal{O}_{\mathbb{R}}} = \mathrm{Re}\,(\omega_{\mathbf{J}} + i\omega_{\mathbf{K}})|_{\mathcal{O}_{\mathbb{R}}} = \omega_{\mathbf{J}}|_{\mathcal{O}_{\mathbb{R}}}$$

The rest is an immediate consequence of our work in §6. Indeed, by Corollary 6.5(iv), $\mathcal{M}(\kappa)^{\gamma_1}$ is a $\mathbf{J}_2$-holomorphic complex submanifold of $\mathcal{M}(\kappa)$. We have $\Phi_1^* \mathbf{J} = \mathbf{J}_2$ by the definition of $\mathbf{J}$. So $\Phi_1(\mathcal{M}(\kappa)^{\gamma_1})$ is a $\mathbf{J}$-complex submanifold of $\mathcal{O}$. But we saw $\Phi_1(\mathcal{M}(\kappa)^{\gamma_1}) = \mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ in (6.12). Thus each connected component of $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ is a $\mathbf{J}$-complex submanifold of $\mathcal{O}$. $\square$

Remark 7.3 Here is another proof that $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ is a $\mathbf{J}$-complex submanifold of $\mathcal{O}$. We can transport the diffeomorphism $\mathcal{R} : \mathcal{M}(\kappa) \to \mathcal{M}(\kappa)$ to a diffeomorphism $\mathcal{R} : \mathcal{O} \to \mathcal{O}$ by means of Φ_1. Then Proposition 6.6 and Corollary 6.8 imply that $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}} = \mathcal{R}(\mathcal{O} \cap \mathfrak{p})$. But $\mathcal{O} \cap \mathfrak{p}$ is an $\mathbf{I}$-complex submanifold of $\mathcal{O}$. So $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$ is a complex submanifold with respect to $\mathcal{R}(\mathbf{I}) = \mathbf{J}$.

Example 7.4 We illustrate Theorem 7.2 in the context of the example of flat hyperkaehler space discussed in Examples 2.1, 2.2, and 4.3. The case $n = 1$ is sufficient to show how this works.

Let $\mathcal{O} \subset \mathfrak{sl}(2, \mathbb{C})$ be the non-zero nilpotent orbit of $SL(2, \mathbb{C})$. We have a covering map

$$\pi : \mathbb{R}^4 - \{0\} \to \mathcal{O}, \qquad \pi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = \begin{pmatrix} uv & -u^2 \\ v^2 & -uv \end{pmatrix}$$

where $u = a + bi$ and $v = c + di$. Let $\mathcal{O}_{\mathbb{R}}$ be the Euclidean connected component of $\mathcal{O} \cap \mathfrak{sl}(2, \mathbb{R})$ containing $\pi(1 + \mathbf{j})$. For $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ we find that

$$\pi(q) \in \mathcal{O}_{\mathbb{R}} \iff u^2, uv, v^2 \in \mathbb{R} \text{ and } u^2 + v^2 > 0 \iff u, v \in \mathbb{R} \iff b = d = 0$$

Thus

$$\pi^{-1}(\mathcal{O}_{\mathbb{R}}) = (\mathbb{R} + \mathbb{R}j) - \{0\}$$

Plainly, $(\mathbb{R} + \mathbb{R}j) - \{0\}$ is a complex submanifold of $\mathbb{H} - \{0\}$ with respect to $\pm\mathbf{J}$ (and only $\pm\mathbf{J}$). But π is a covering of hyperkaehler manifolds, and so it follows that $\mathcal{O}_{\mathbb{R}}$ is a complex submanifold of $\mathbb{H} - \{0\}$ with respect to $\pm\mathbf{J}$ (and only $\pm\mathbf{J}$).

We have a source of $\mathbf{J}$-holomorphic functions on $\mathcal{O}$, namely the component functions $\Phi_{\mathbf{J}}^z$, $z \in \mathfrak{g}$, of the $\mathbf{J}$-holomorphic moment map $\Phi_{\mathbf{J}} : \mathcal{O} \to \mathfrak{g}$. We next examine how these restrict to $\mathcal{O}_{\mathbb{R}}$. For this, we recall the complex Cartan decomposition (7.9).

Theorem 7.5 *The functions* $\Phi_{\mathbf{J}}^x$, $x \in \mathfrak{k}$, *vanish identically on* $\mathcal{O}_{\mathbb{R}}$. *However for* $v \in \mathfrak{p}$, *the functions*

$$f^v = i\Phi_{\mathbf{J}}^v|_{\mathcal{O}_{\mathbb{R}}} \tag{7.13}$$

separate the points of $\mathcal{O}_\mathbb{R}$ and the corresponding holomorphic map $\mathcal{V} : \mathcal{O}_\mathbb{R} \to \mathfrak{p}$ defined by the f^v, so that $(v, \mathcal{V}(w))_\mathfrak{g} = f^v(w)$, is a locally closed embedding.

The image $Y = \mathcal{V}(\mathcal{O}_\mathbb{R})$ is a single K-orbit in $\mathfrak{p}$. Y is a connected component of $\mathcal{O} \cap \mathfrak{p}$ and so in particular Y is stable under the dilation action of $\mathbb{C}^*$. The resulting $(K_\mathbb{R} \times \mathbb{R}^+)$-equivariant diffeomorphism

$$\mathcal{V} : \mathcal{O}_\mathbb{R} \to Y \tag{7.14}$$

is the Vergne diffeomorphism.

Proof We know by Proposition 6.6 that $\Phi_\mathbf{J}$ gives a diffeomorphism of $\mathcal{O} \cap \mathfrak{g}_\mathbb{R}$ onto $\mathcal{O} \cap \mathfrak{p}$. This means that $\Phi_\mathbf{J}^x$, $x \in \mathfrak{k}$, vanishes on $\mathcal{O} \cap \mathfrak{g}_\mathbb{R}$ while the functions $\Phi_\mathbf{J}^v$, $v \in \mathfrak{p}$, embed $\mathcal{O} \cap \mathfrak{g}_\mathbb{R}$ into $\mathfrak{p}$ as $\mathcal{O} \cap \mathfrak{p}$. But then the functions $i\Phi_\mathbf{J}^v$ do the same thing. The diffeomorphism

$$\mathcal{V} : \mathcal{O} \cap \mathfrak{g}_\mathbb{R} \to \mathcal{O} \cap \mathfrak{p} \tag{7.15}$$

defined by $i\Phi_\mathbf{J}$ is the Vergne diffeomorphism discovered in [Ve].

Now $Y = \mathcal{V}(\mathcal{O}_\mathbb{R})$ is a connected component of $\mathcal{O} \cap \mathfrak{p}$. We know by a well-known argument that Y is K-homogeneous. Indeed, K acts on $\mathcal{O} \cap \mathfrak{p}$ and at any point $e \in Y$, we have a complex Lagrangian decomposition

$$T_e\mathcal{O} = [\mathfrak{g}, e] = [\mathfrak{k}, e] \oplus [\mathfrak{p}, e] \tag{7.16}$$

Also $[\mathfrak{k}, e] \subset \mathfrak{p}$ while $[\mathfrak{p}, e] \subset \mathfrak{k}$. It follows that $T_eY = [\mathfrak{k}, e]$. So the K-action on Y is infinitesimally transitive and hence transitive. $\square$

Remark 7.6 (i) We introduced the factor i in (7.13) for convenience (later in Theorems 7.9 and 9.3) Essentially, the factor i arises because it is $i\mathcal{O}_\mathbb{R}$, not $\mathcal{O}_\mathbb{R}$, which most naturally corresponds to Y in the Sekiguchi correspondence.

(ii) The Vergne diffeomorphism in (7.15) recovers and explains the Kostant-Sekiguchi correspondence [Se].

Corollary 7.7 *The $K_\mathbb{R}$-action on $\mathcal{O}_\mathbb{R}$ complexifies with respect to $\mathbf{J}$ to give a transitive holomorphic action of K on $\mathcal{O}_\mathbb{R}$. Then the Vergne diffeomorphism (7.14) is K-equivariant.*

Theorem 7.5 and Proposition 6.6 give

Corollary 7.8 *[Ve] Let $A \in \mathcal{M}(\kappa)$. Then*

$$\Phi_1(A) \in \mathcal{O}_\mathbb{R} \quad \Longleftrightarrow \quad \Phi_2(A) \in Y \quad \Longleftrightarrow \quad \Phi_3(A) \in i\mathcal{O}_\mathbb{R} \tag{7.17}$$

Let

$$\mu : \mathcal{O}_\mathbb{R} \to \mathfrak{k}_\mathbb{R} \tag{7.18}$$

be the projection map defined by the Cartan decomposition (7.7). Then μ is a $K_\mathbb{R}$-equivariant moment map for the $K_\mathbb{R}$-action (inside the $G_\mathbb{R}$-action) on $\mathcal{O}$.

Theorem 7.9 *Triple Sum Formula. Let $w \in \mathcal{O}_\mathbb{R}$. Then the Vergne diffeomorphism (7.14) satisfies*

$$w = \mu(w) + \frac{1}{2}\mathcal{V}(w) + \frac{1}{2}\overline{\mathcal{V}(w)} \tag{7.19}$$

In (7.19), we are taking the sum inside $\mathfrak{g}$ of the three vectors. In view of (7.7), (7.19) says that $\frac{1}{2}\mathcal{V}(w) + \frac{1}{2}\overline{\mathcal{V}(w)}$ is the projection of w to $\mathfrak{p}_\mathbb{R}$.

Proof Given $w \in \mathcal{O}_\mathbb{R}$, let $A \in \mathcal{M}(\kappa)$ be the (unique) instanton such that

$$w = \Phi_1(A) = \zeta_2(A) + i\zeta_3(A) \tag{7.20}$$

By Proposition 6.6, we have (taking $\mathfrak{a} = \mathfrak{k}_\mathbb{R}$, $\mathfrak{b} = i\mathfrak{p}_\mathbb{R}$)

$$\zeta_1(A) \in i\mathfrak{p}_\mathbb{R}, \qquad \zeta_2(A) \in \mathfrak{k}_\mathbb{R}, \qquad \zeta_3(A) \in i\mathfrak{p}_\mathbb{R} \tag{7.21}$$

So the projection of w to $\mathfrak{k}_\mathbb{R}$ is

$$\mu(w) = \zeta_2(A) \tag{7.22}$$

Now

$$\mathcal{V}(w) = i\Phi_2(A) = i(\zeta_3(A) + i\zeta_1(A)) = -\zeta_1(A) + i\zeta_3(A) \tag{7.23}$$

Since $\zeta_1(A) \in i\mathfrak{p}_\mathbb{R}$ is pure imaginary and $i\zeta_3(A) \in \mathfrak{p}_\mathbb{R}$ is real, we get

$$\overline{\mathcal{V}(w)} = \zeta_1(A) + i\zeta_3(A) \tag{7.24}$$

So

$$\frac{1}{2}\left(\mathcal{V}(w) + \overline{\mathcal{V}(w)}\right) = i\zeta_3(A) \tag{7.25}$$

Now (7.19) is immediate. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The real symmetric pair $(\mathfrak{g}_\mathbb{R}, \mathfrak{k}_\mathbb{R})$ (or the complex symmetric pair $(\mathfrak{g}, \mathfrak{k})$) is called *Hermitian* if the corresponding real symmetric space $G_\mathbb{R}/K_\mathbb{R}$ has a $K_\mathbb{R}$-invariant Hermitian structure. This amounts to the condition that we can find $x_0 \in \mathrm{Cent}\,\mathfrak{k}_\mathbb{R}$ such that the eigenvalues of $\mathrm{ad}\,x_0$ on $\mathfrak{p}$ are $\pm i$. Then we get a $K_\mathbb{R}$-stable splitting

$$\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$$

where $\mathfrak{p}^\pm$ is the $\pm i$-eigenspace. Then $\dim_\mathbb{C}\mathfrak{p}^+ = \dim_\mathbb{C}\mathfrak{p}^-$ and moreover $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are mutually contragredient $K_\mathbb{R}$-representations.

Lemma 7.10 *Suppose $(\mathfrak{g}, \mathfrak{k})$ is Hermitian. Then the following two $\mathbb{R}$-linear maps are inverses:*

$$\mathfrak{p}_\mathbb{R} \overset{\iota}{\longrightarrow} \mathfrak{p}^+, \quad u \mapsto \frac{1}{2}\left(u - i[x_0, u]\right) \qquad \text{and} \qquad \mathfrak{p}^+ \to \mathfrak{p}_\mathbb{R}, \quad v \mapsto v + \overline{v}$$

The Addition Formula recovers in the Hermitian case the fact

Corollary 7.11 *[Ve, Prop. 6] Suppose $(\mathfrak{g}, \mathfrak{k})$ is Hermitian and $\mathcal{O}_\mathbb{R}$ is such that $Y = \mathcal{V}(\mathcal{O}_\mathbb{R})$ lies in $\mathfrak{p}^+$. Then the Vergne diffeomorphism $\mathcal{V} : \mathcal{O}_\mathbb{R} \to Y$ is given by the composition*

$$\mathcal{O}_\mathbb{R} \hookrightarrow \mathfrak{g}_\mathbb{R} \to \mathfrak{p}_\mathbb{R} \overset{\iota}{\longrightarrow} \mathfrak{p}^+$$

where the middle map is the projection defined by the Cartan decomposition (7.7).

8 The KV Circle Action on $\mathcal{O}_\mathbb{R}$ and the Kaehler Potential ρ_o

The instanton Kaehler structure $(\mathbf{J}, \sigma)$ We let $\mathrm{d} = \partial + \overline{\partial}$ be the canonical splitting of d into its $\mathbf{J}$-holomorphic and $\mathbf{J}$-antiholomorphic parts. (So $\partial = \partial_\mathbf{j}$ and $\overline{\partial} = \overline{\partial}_\mathbf{j}$.)

Corollary 5.4 and Corollary 5.2(ii) give

Corollary 8.1 *The complex nilpotent orbit $\mathcal{O}$, equipped with its U-invariant instanton hyperkaehler structure, admits a U-invariant hyperkaehler potential $\rho : \mathcal{O} \to \mathbb{R}$, unique up to addition of a constant. The further condition that ρ is homogeneous of degree 1, with respect to the Euler scaling action of $\mathbb{R}^+$ on $\mathcal{O}$, determines ρ uniquely.*

Let

$$\rho_o : \mathcal{O}_\mathbb{R} \to \mathbb{R} \tag{8.1}$$

be the smooth function obtained by restricting the hyperkaehler potential ρ from Corollary 8.1. Plainly Theorem 5.1 and Corollary 8.1 give

Corollary 8.2 *ρ_o is a global Kaehler potential on $\mathcal{O}_\mathbb{R}$ so that*

$$\sigma = i\partial\overline{\partial}\rho_o \tag{8.2}$$

The function ρ_o is $K_\mathbb{R}$-invariant and homogeneous of degree 1 with respect to the Euler scaling action of $\mathbb{R}^+$ on $\mathcal{O}_\mathbb{R}$.

Next we consider how the Kronheimer $SO(3)$-action on $\mathcal{O}$ from §7 transforms the submanifold $\mathcal{O}_\mathbb{R}$.

Proposition 8.3 *The circle subgroup $\mathcal{C}_\mathbf{j}$ of $SO(3)$ preserves $\mathcal{O}_\mathbb{R}$ and so defines a smooth group action*

$$\Pi : S^1 \to \mathrm{Diff}\,\mathcal{O}_\mathbb{R} \tag{8.3}$$

We call this the KV (Kronheimer-Vergne) S^1-action, and we let θ denote the infinitesimal generator.

The KV S^1-action on $\mathcal{O}_\mathbb{R}$ is free, commutes with the $K_\mathbb{R}$-action and is Kaehler. Moreover the KV S^1-action is Hamiltonian and ρ_o is an equivariant moment map. I.e.,

$$\theta \lrcorner \sigma + \mathrm{d}\rho_o = 0 \tag{8.4}$$

or, equivalently, The Hamiltonian flow of ρ_o is the KV S^1-action.

We write the KV S^1-action as, for $w \in \mathcal{O}_\mathbb{R}$,

$$\Pi(\mathrm{e}^{it})(w) = \mathrm{e}^{it} \star w \tag{8.5}$$

Proof The first statement is clear by, e.g., Corollary 6.4. The properties of the KV S^1-action are then immediate from the corresponding properties of the Kronheimer $SO(3)$-action on $\mathcal{O}$ or $\mathcal{M}(\kappa)$. $\qquad\qquad\square$

The $\mathbf{J}$-holomorphic functions $\Phi^z_\mathbf{j}$, $z \in \mathfrak{g}$, on $\mathcal{O}$ transform under the action of $\mathcal{C}_\mathbf{j}$ by the degree 1 character. Consequently Proposition 8.3 gives

Corollary 8.4 *For $v \in \mathfrak{p}$, the **J**-holomorphic functions f^v on $\mathcal{O}_{\mathbb{R}}$ satisfy*

$$\{\rho_o, f^v\} = if^v \tag{8.6}$$

We get insight into the KV S^1-action by trying to find it inside the action of $K_{\mathbb{R}}$. For simplicity of exposition, we assume in the next result that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is simple, then either

(I) $(\mathfrak{g}, \mathfrak{k})$ is Hermitian and $\mathrm{Cent}\, \mathfrak{k}_{\mathbb{R}} = \mathbb{R}x_0$ where $x_0 \neq 0$ was chosen in §7.9, or

(II) $(\mathfrak{g}, \mathfrak{k})$ is non-Hermitian, $\mathrm{Cent}\, \mathfrak{k}_{\mathbb{R}} = 0$ and $\mathfrak{p}$ is irreducible as a $K_{\mathbb{R}}$-representation.

Proposition 8.5 *Suppose $\mathfrak{g}$ is a simple complex Lie algebra. Then*

(I) *If $(\mathfrak{g}, \mathfrak{k})$ is Hermitian and $Y \subset \mathfrak{p}^{\pm}$ then the KV S^1-action on $\mathcal{O}_{\mathbb{R}}$ is given by the center of $K_{\mathbb{R}}$ and*

$$\rho_o = \phi^{\mp x_0} \tag{8.7}$$

(II) *Otherwise, the KV S^1-action on $\mathcal{O}_{\mathbb{R}}$ lies outside the action of $G_{\mathbb{R}}$.*

Proof To prove that $\rho_o = \phi^{cx_0}$ where c is constant, it is necessary and sufficient to show that

$$\{\rho_o, f^v\} = \{\phi^{cx_0}, f^v\} \tag{8.8}$$

for all $v \in \mathfrak{p}$. This is because the functions f^v, $v \in \mathfrak{p}$, separate the points of $\mathcal{O}_{\mathbb{R}}$ and both ρ_o and ϕ^{x_0} are $\mathbb{R}^+$-homogeneous of degree 1.

Let $v \in \mathfrak{p}$. Then $\{\phi^{x_0}, f^v\} = f^{[x_0, v]}$ and so in the Hermitian case we get

$$\{\phi^{x_0}, f^v\} = \begin{cases} if^v & \text{if } v \in \mathfrak{p}^+ \\ -if^v & \text{if } v \in \mathfrak{p}^- \end{cases} \tag{8.9}$$

We want to compare this with (8.6). If $Y \subset \mathfrak{p}^+$, then f^v vanishes on Y for all $v \in \mathfrak{p}^+$. For $v \in \mathfrak{p}^-$, (8.8) holds with $c = -1$. So then $\rho_o = \phi^{-x_0}$. Similarly $Y \subset \mathfrak{p}^-$ gives $\rho_o = \phi^{x_0}$. In either case, it follows that $\mathrm{Cent}\, K_{\mathbb{R}}$ gives the Hamiltonian flow of ρ_o, and this is just the KV circle action.

Now, continuing the Hermitian case, suppose Y fails to lie in $\mathfrak{p}^+$ or $\mathfrak{p}^-$. Then we cannot find $c \in \mathbb{R}$ such that (8.8) holds for all $v \in \mathfrak{p}$. Since ρ_o is $K_{\mathbb{R}}$-invariant and (up to scaling) ϕ^{x_0} is the only $K_{\mathbb{R}}$-invariant function in $\phi^*(\mathfrak{g}_{\mathbb{R}})$, we see that ρ_o lies outside of $\phi^*(\mathfrak{g}_{\mathbb{R}})$. If $(\mathfrak{g}, \mathfrak{k})$ is non-Hermitian, then $\phi^*(\mathfrak{g}_{\mathbb{R}})$ has no non-zero $K_{\mathbb{R}}$-invariants, and so certainly ρ_o lies outside of $\phi^*(\mathfrak{g}_{\mathbb{R}})$.

The condition that ρ_o lies outside of $\phi^*(\mathfrak{g}_{\mathbb{R}})$ means exactly that the Hamiltonian flow of ρ_o lies outside the action of $G_{\mathbb{R}}$. $\square$

Using ρ_o, we next prove in Theorem 8.6 below that the Vergne diffeomorphism produces an embedding of $\mathcal{O}_{\mathbb{R}}$ into T^*Y which realizes T^*Y as a symplectic complexification of $\mathcal{O}_{\mathbb{R}}$.

Let

$$s : \mathcal{O}_{\mathbb{R}} \xrightarrow{\mathcal{V} \times \mu} Y \times \mathfrak{k}_{\mathbb{R}} \longrightarrow Y \times \mathfrak{k} \tag{8.10}$$

be the smooth map built out of the Vergne diffeomorphism, the moment map μ and the obvious inclusion $\mathfrak{k}_{\mathbb{R}} \hookrightarrow \mathfrak{k}$. Then s is an embedding of $\mathcal{O}_{\mathbb{R}}$ into $Y \times \mathfrak{k}$, since already $\mathcal{V}$ is 1-to-1.

On the other hand, we have a natural holomorphic embedding

$$T^*Y \to Y \times \mathfrak{k} \tag{8.11}$$

of the cotangent bundle T^*Y of Y. Indeed, differentiating the K-action on Y we get an infinitesimal holomorphic vector field action

$$\mathfrak{k} \to \mathfrak{Vect}^{hol} \, \mathcal{O}_{\mathbb{R}}, \qquad x \mapsto \eta^x \tag{8.12}$$

The induced K-action on T^*Y is holomorphic Hamiltonian with momentum functions given by the holomorphic symbols of the vector fields η^x. Let $M : T^*Y \to \mathfrak{k}$ be the corresponding holomorphic moment map. Then (8.11) is the product of the canonical projection $T^*Y \to Y$ with M. It follows since the K-action on $\mathcal{O}_{\mathbb{R}}$ is transitive that the product map is 1-to-1.

Theorem 8.6 *The image of s in $Y \times \mathfrak{k}$ lies inside T^*Y. Furthermore the resulting $K_{\mathbb{R}}$-equivariant map*

$$s : \mathcal{O}_{\mathbb{R}} \to T^*Y \tag{8.13}$$

embeds $(\mathcal{O}_{\mathbb{R}}, \sigma)$ as a totally real symplectic submanifold of (T^*Y, Ω) so that

$$s^*(\mathrm{Re}\,\Omega) = \sigma \qquad and \qquad s^*(\mathrm{Im}\,\Omega) = 0 \tag{8.14}$$

where Ω is the canonical holomorphic symplectic form on T^*Y.

Proof We may regard the vector fields η^x on Y as vector fields on $\mathcal{O}_{\mathbb{R}}$ using the Vergne diffeomorphism. Showing that the image of s lies in T^*Y amounts to showing that $\mu : \mathcal{O}_{\mathbb{R}} \to \mathfrak{k}_{\mathbb{R}}$ is given by a real smooth 1-form β on $\mathcal{O}_{\mathbb{R}}$ in the sense that for all $x \in \mathfrak{k}_{\mathbb{R}}$ we have

$$\langle \beta, \eta^x \rangle = \mu^x \qquad and \qquad \langle \beta, \mathbf{J}\eta^x \rangle = 0 \tag{8.15}$$

This follows easily because, since K acts transitively on $\mathcal{O}_{\mathbb{R}}$ by Corollary 7.7, the holomorphic tangent spaces of $(\mathcal{O}_{\mathbb{R}}, \mathbf{J})$ are spanned by the vector fields η^x, $\mathbf{J}\eta^x$, $x \in \mathfrak{k}_{\mathbb{R}}$.

Furthermore, we find a 1-form β giving μ, then

$$s^*(\mathrm{Re}\,\Theta) = \beta \tag{8.16}$$

where Θ is the holomorphic canonical 1-form on T^*Y. Then $\mathrm{d}\Theta = \Omega$ and so (8.7.5) holds if and only if $\mathrm{d}\beta = \sigma$, i.e., if and only if β is a symplectic potential.

So the problem is to produce a symplectic potential β on $\mathcal{O}_{\mathbb{R}}$ satisfying (8.15). We claim that

$$\beta = -\frac{i}{2}(\partial - \overline{\partial})\rho_o \tag{8.17}$$

works. First β is a symplectic potential since

$$\mathrm{d}\beta = -\frac{i}{2}(\partial + \overline{\partial})(\partial - \overline{\partial})\rho_o = i\partial\overline{\partial}\rho_o = \sigma \tag{8.18}$$

It follows that the functions $\langle \beta, \eta^x \rangle$ are momentum functions for the $K_{\mathbb{R}}$-action on $\mathcal{O}_{\mathbb{R}}$. Consequently, for each $x \in \mathfrak{k}_{\mathbb{R}}$, $\langle \beta, \eta^x \rangle$ is equal to μ^x up to addition of a constant. But both

functions $\langle \beta, \eta^x \rangle$ and μ^x are homogeneous of degree 1 with respect to the Euler $\mathbb{R}^+$-action, and so $\langle \beta, \eta^x \rangle = \mu^x$. Finally, we find

$$\langle \beta, \mathbf{J}\eta^x \rangle = \langle \mathbf{J}\beta, \eta^x \rangle = \frac{1}{2}\langle (\partial + \overline{\partial})\rho_o, \eta^x \rangle = \frac{1}{2}\langle \mathrm{d}\rho_o, \eta^x \rangle = \frac{1}{2}\eta^x(\rho_o) = 0 \qquad (8.19)$$

since $\mathbf{J}\partial = i\partial$, $\mathbf{J}\overline{\partial} = -i\overline{\partial}$ and ρ_o is $K_{\mathbb{R}}$-invariant. This proves (8.14) and it follows easily that $\mathcal{O}_{\mathbb{R}}$ is a totally real submanifold of T^*Y. $\hspace{2cm}\square$

Corollary 8.7 *The embedding s realizes (T^*Y, Ω) as a symplectic complexification of $(\mathcal{O}_{\mathbb{R}}, \sigma)$.*

9   Toward Geometric Quantization of Real Nilpotent Orbits

We can interpret Theorem 7.2 as providing a polarization on $\mathcal{O}_{\mathbb{R}}$ in the sense of Geometric Quantization.

Corollary 9.1 $\mathcal{O}_{\mathbb{R}}$, *equipped with its KKS symplectic form σ, admits two transverse $K_{\mathbb{R}}$-invariant complex polarizations, namely the holomorphic and anti-holomorphic tangent spaces defined by the instanton Kaehler structure.*

Remark 9.2 We have two rather different extensions of the Kaehler $K_{\mathbb{R}}$-action on $\mathcal{O}_{\mathbb{R}}$ to a transitive action of a larger group. These extensions are given by the $G_{\mathbb{R}}$-action and the K-action. But neither of these larger actions is Kaehler. Indeed the $G_{\mathbb{R}}$-action is symplectic but not holomorphic, while the K-action is holomorphic but not symplectic.

The instanton Kaehler structure on $\mathcal{O}_{\mathbb{R}}$ provides the first step in our quantization program for $\mathcal{O}_{\mathbb{R}}$. In the spirit of Geometric Quantization (GQ), the quantization problem on $\mathcal{O}_{\mathbb{R}}$ is to convert $\mathbb{R}$-valued smooth functions ϕ on $\mathcal{O}_{\mathbb{R}}$ into Hermitian operators $\mathcal{Q}(\phi)$ on a Hilbert space $\mathcal{H}$ of $\mathbf{J}$-holomorphic half-forms on $\mathcal{O}_{\mathbb{R}}$. This conversion must satisfy the Dirac axiom that the Poisson bracket of functions goes over into the commutator of operators so that

$$\mathcal{Q}\left(\{\phi, \psi\}\right) = i[\mathcal{Q}(\phi), \mathcal{Q}(\psi)] \qquad (9.1)$$

By the well-known No-Go Theorem, this conversion cannot be carried out for all smooth functions (or even for all polynomial functions on a real symplectic vector space) consistently so that (9.1) is satisfied. However, we expect (for various reasons) that the Hamiltonian functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, in (7.11) *can be* quantized consistently, modulo some "isolated anomalous" cases.

Quantization of the functions ϕ^w, $w \in \mathfrak{g}_{\mathbb{R}}$, already would solve the Orbit Method problem in representation theory of attaching to $\mathcal{O}_{\mathbb{R}}$ an irreducible unitary representation (or finitely many such representations) of the universal cover of $G_{\mathbb{R}}$. Indeed, the operators $i\mathcal{Q}(\phi^x)$ define a Lie algebra representation of $\mathfrak{g}_{\mathbb{R}}$ on the space of $K_{\mathbb{R}}$-finite vectors in $\mathcal{H}$.

The larger GQ problem is to construct the operators $\mathcal{Q}(\phi^x)$ and the Hermitian positive definite inner product on (some subspace) of holomorphic half-forms that gives rise to $\mathcal{H}$.

The $K_{\mathbb{R}}$-invariance of the instanton Kaehler structure means that the Hamiltonian flow of the functions ϕ^x, $x \in \mathfrak{k}_{\mathbb{R}}$ preserves $\mathbf{J}$. Thus the Lie derivative of the Hamiltonian vector field $\xi_{\phi^x} = \eta^x$ gives us a natural choice for the quantization, namely

$$\mathcal{Q}(\phi^x) = -i\mathcal{L}_{\eta^x}, \qquad x \in \mathfrak{k}_{\mathbb{R}} \qquad (9.2)$$

But the Hamiltonian flow of the "remaining" functions ϕ^v, $v \in \mathfrak{p}_\mathbb{R}$, does not preserve $\mathbf{J}$. Our strategy for quantizing these remaining functions is to "decompose" them in terms of holomorphic and anti-holomorphic functions on $\mathcal{O}_\mathbb{R}$. The reason for this is that we expect that a holomorphic function f quantizes to multiplication by g, while an anti-holomorphic function $\overline{g}$ quantizes to the adjoint of multiplication by f. (Another Dirac axiom is that $\mathcal{Q}(f)$ and $\mathcal{Q}(\overline{f})$ are adjoint.) Here $\mathcal{Q}(f) = f$ and $\mathcal{Q}(\overline{g}) = \mathcal{Q}(g)^*$ are densely defined operators on $\mathcal{H}$.

A decomposition of ϕ^v with respect to holomorphic and anti-holomorphic functions should be of the form $\phi^v = \sum f_p \overline{g}_p$ where f_p and g_p are holomorphic functions. Of course, the sum could well be infinite and unwieldy to quantize.

The key point is that the decomposition of ϕ^v is remarkably simple:

Theorem 9.3 *Let $v \in \mathfrak{p}_\mathbb{R}$. Then the Hamiltonian function $\phi^v : \mathcal{O}_\mathbb{R} \to \mathbb{R}$ is the real part of a $\mathbf{J}$-holomorphic function on $\mathcal{O}_\mathbb{R}$. In fact*

$$\phi^v = \operatorname{Re} f^v = \frac{1}{2}\left(f^v + \overline{f^v}\right) \tag{9.3}$$

where f^v is the $\mathbf{J}$-holomorphic function defined in Theorem 7.5. Moreover

$$f^v = \phi^v - i\{\rho_o, \phi^v\} \tag{9.4}$$

Proof The first part is a corollary of the Triple Sum Formula Theorem 7.9. Indeed, let $w \in \mathcal{O}_\mathbb{R}$. Then using (7.19) we find

$$
\begin{aligned}
2\phi^v(w) &= 2(v, w)_\mathfrak{g} = (v, \mathcal{V}(w) + \overline{\mathcal{V}(w)})_\mathfrak{g} = (v, \mathcal{V}(w))_\mathfrak{g} + \overline{(v, \mathcal{V}(w))_\mathfrak{g}} \\
&= f^v(w) + \overline{f^v(w)}
\end{aligned}
$$

So $\phi^v = \frac{1}{2}(f^v + \overline{f^v})$.

Next, we have $\{-i\rho_o, f^v\} = f^v$ by (8.6) and so $\{-i\rho_o, \overline{f^v}\} = -\overline{f^v}$ by complex conjugation. Then (9.3) gives

$$\{-i\rho_o, \phi^v\} = \frac{1}{2}(f^v - \overline{f^v}) \tag{9.5}$$

Now adding (9.3) and (9.5) we get (9.4). □

We may choose a set $z_1, \ldots, z_n$ of local holomorphic coordinates on $\mathcal{O}_\mathbb{R}$. Then ϕ^v is a real analytic function in the $2n$ coordinates $z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n$ and (9.3) says that

$$\phi^v(z_1, \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n) = f_v(z_1, \ldots, z_n) + \overline{f_v}(\overline{z}_1, \ldots, \overline{z}_n)$$

Corollary 9.4 *The functions ϕ^v, $v \in \mathfrak{p}_\mathbb{R}$, on $\mathcal{O}_\mathbb{R}$ are pluriharmonic.*

Remark 9.5 (i) Alternatively, we can prove (9.3) right after Theorem 7.2 in the following way. We find for $v \in \mathfrak{p}_\mathbb{R}$

$$\phi^v = \operatorname{Re} \Phi_\mathbf{I}^v = \zeta_\mathbf{K}^{iv}$$

because (2.16) gives $\Phi_{\mathbf{I}}^v = -i\Phi_{\mathbf{I}}^{iv} = -i\zeta_{\mathbf{J}}^{iv} + \zeta_{\mathbf{K}}^{iv}$. But also we have the $\mathbf{J}$-holomorphic function

$$i\Phi_{\mathbf{J}}^v = \Phi_{\mathbf{J}}^{iv} = \zeta_{\mathbf{K}}^{iv} + i\zeta_{\mathbf{I}}^{iv}$$

So $\phi^v = \operatorname{Re} f^v$ where $f^v = i\Phi_{\mathbf{J}}^v$.

(ii) The condition $\phi^v = \operatorname{Re} F^v$ determines a holomorphic function F^v uniquely up to addition of a constant. The additional condition that F^v is homogeneous then forces $F^v = f^v$.

We summarize the way the KV S^1-action produces the complex structure $\mathbf{J}$ on $\mathcal{O}_{\mathbb{R}}$ in the next Corollary. Let

$$\mathfrak{v} = \phi^*(\mathfrak{p}_{\mathbb{R}}) = \{\phi^v \,|\, v \in \mathfrak{p}_{\mathbb{R}}\} \tag{9.6}$$

so that $\mathfrak{v}$ is the space of Hamiltonian functions on $\mathcal{O}_{\mathbb{R}}$ corresponding to $\mathfrak{p}_{\mathbb{R}}$.

Corollary 9.6 *Let $\mathfrak{v}^{\sharp} \subset C^{\infty}(\mathcal{O}_{\mathbb{R}})$ be the subspace spanned by all the translates of $\mathfrak{v}$ under the KV S^1-action. If $\mathfrak{g}$ is simple then in Cases (I) and (II) of Proposition 8.5 we find*

(I) $\mathfrak{v}^{\sharp} = \mathfrak{v}$ *and so $\mathfrak{v} \simeq \mathfrak{p}_{\mathbb{R}}$ as $K_{\mathbb{R}}$-representations.*

(II) *We have the direct sum*

$$\mathfrak{v}^{\sharp} = \mathfrak{v} \oplus \{\rho_o, \mathfrak{v}\} \tag{9.7}$$

and so $\mathfrak{v} \simeq \mathfrak{p}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ as $K_{\mathbb{R}}$-representations.

In either case, $\mathfrak{v}^{\sharp}$ decomposes under the KV action of $S^1 \simeq SO(2)$ into a direct sum of copies of the 2-dimensional rotation representation so that $\mathfrak{v} \simeq \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^2$. Consequently the complexification $\mathfrak{v}_{\mathbb{C}}$ splits into the direct sum

$$\mathfrak{v}_{\mathbb{C}} = \mathfrak{v}_{\mathbb{C}}^+ \oplus \mathfrak{v}_{\mathbb{C}}^- \tag{9.8}$$

where $\mathfrak{v}_{\mathbb{C}}^{\pm} = \{\phi \in \mathfrak{v} \,|\, e^{i\theta} \star \phi = e^{\pm i\theta}\phi\}$. In Case I, $\mathfrak{v}_{\mathbb{C}}^{\pm}$ identifies with $\mathfrak{p}^{\pm}$.

In either case, $\mathfrak{v}_{\mathbb{C}}^+$ and $\mathfrak{v}_{\mathbb{C}}^-$ are complex-conjugate $K_{\mathbb{R}}$-stable spaces of complex-valued Poisson commuting functions on $\mathcal{O}_{\mathbb{R}}$. We have

$$\mathfrak{v}_{\mathbb{C}}^+ = \{f^v \,|\, v \in \mathfrak{p}\} \qquad and \qquad \mathfrak{v}_{\mathbb{C}}^- = \{\overline{f^v} \,|\, v \in \mathfrak{p}\} \tag{9.9}$$

Finally $\mathbf{J}$ is the unique complex structure on $\mathcal{O}_{\mathbb{R}}$ such that the functions f^v, $v \in \mathfrak{p}$, are $\mathbf{J}$-holomorphic.

Remark 9.7 Corollary 9.6 says in particular that

$$\mathfrak{v}^{\sharp} = \mathfrak{v} + \{\rho_o, \mathfrak{v}\} + \{\rho_o, \{\rho_o, \mathfrak{v}\}\} + \cdots$$

Moreover, we could take this as the definition of $\mathfrak{v}^{\sharp}$, and then all the assertions in Corollary 9.6 read the same.

A   Appendix: Real Algebraic Varieties and Nash Manifolds

A.1   Introduction

In this appendix we present some basic notions from the theory of real algebraic varieties. Some references are [BoCR], [BeR], [BoE].

On many points, we follow the treatment in [BoCR]. We add material about real algebraic variety structures associated to complex algebraic varieties. Our discussion of Nash manifolds and Nash functions is more general. The last subsection §A.10 explains the application of this theory to real groups and their orbits.

We present this material here for lack of an appropriate reference in the literature.

A.2   Real Algebraic Sets

A subset $V \subset \mathbb{R}^n$ is a (real) *algebraic set* if V is the set of common zeroes of some finite set of real polynomial functions $P \in \mathbb{R}[x_1, \ldots, x_n]$. Then a function $f : V \to \mathbb{R}$ is called *regular* if there exist polynomial functions $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ such that Q has no zeroes on V and $f(x) = P(x)/Q(x)$ for all $x \in V$. The set of regular functions on V forms an $\mathbb{R}$-algebra which we will call $A(V)$.

Let $P(V) \subset A(V)$ be the image of the natural ring homomorphism $\mathbb{R}[x_1, \ldots, x_n] \to A(V)$. Then

$$P(V) \simeq \mathbb{R}[x_1, \ldots, x_n]/I(V)$$

where $I(V) \subset P(V)$ is the ideal of polynomial functions vanishing on V. Notice that $A(V)$ is algebraic over $P(V)$; indeed, if $f = P/Q$ then $Qf - P = 0$.

A map $\phi : V \to V'$, where $V' \subset \mathbb{R}^m$ is an algebraic set, is *regular* if each of the component functions $\phi_1, \ldots, \phi_m$ is regular. An equivalent condition is that the pullback of a regular function on V' is regular on V; then ϕ induces an algebra homomorphism $\phi^* : A(V') \to A(V)$. Conversely, any algebra homomorphism $p : A(V') \to A(V)$ defines uniquely a regular map $V \to V'$.

An isomorphism of algebraic sets is a bijective biregular map. Isomorphisms $V \to V'$ are in natural bijection with algebra isomorphisms $A(V') \to A(V)$.

We may also use the term *real algebraic* in speaking of regular functions and maps.

A.3   Real Affine Algebraic Varieties

The *Zariski topology* on an algebraic set $V \subset \mathbb{R}^n$ is defined just as in the complex case, so that the Zariski closed sets in V are precisely the algebraic sets in $\mathbb{R}^n$ which lie in V. This topology is not Hausdorff but it is Noetherian and hence quasi-compact (every open cover has a finite subcover) as the polynomial ring $R[x_1, \ldots, x_n]$ is Noetherian.

Every Zariski closed set is closed in the usual Euclidean topology on $\mathbb{R}^n$ defined by the Euclidean metric, as polynomials are continuous. We will refer to open sets, closed sets, etc as "Zariski" or "Euclidean" to distinguish the two topologies.

A topological space M is called irreducible if M cannot be written as the union of two closed subsets different from M. We say an algebraic set is *irreducible* if it is irreducible in the Zariski topology.

A *regular function* on a Zariski open set $U \subset V$ is one of the form $P(x)/Q(x)$ where $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ and Q is nowhere vanishing on U. The set of regular functions on U is closed under composition and forms an $\mathbb{R}$-algebra which we will denote $A_V(U)$.

The assignment $U \mapsto A_V(U)$ defines a sheaf $\mathcal{A}_V$ of $\mathbb{R}$-algebras on V with respect to its Zariski topology. In particular, if $U_1, \ldots, U_m$ is a finite Zariski open cover of a Zariski open set $U \subset V$ and f is a real-valued function on U such that $f|_{U_i} = P_i/Q_i$ then we can find $P, Q \in \mathbb{R}[x_1, \ldots, x_n]$ such that Q is nowhere vanishing on U and $f = P/Q$. Indeed, as $U_i \subset V$ is open, the complement $V - U_i$ is the zero-locus of a finite set of polynomials; let F_i be the sum of their squares. Then $U_i = V \cap (F_i \neq 0)$ and the polynomials $P = \sum_i^m P_i Q_i F_i^2$ and $Q = \sum_i^m Q_i^2 F_i^2$ satisfy our requirement.

Then $(V, \mathcal{A}_V)$ is a ringed space in the usual sheaf theory sense.

Now we can define an (abstract) *real affine algebraic variety*: this is a pair $(X, \mathcal{A}_X)$ where X is an irreducible topological space, $\mathcal{A}_X$ is a sheaf of $\mathbb{R}$-algebras of $\mathbb{R}$-valued functions on X and there exists an isomorphism of ringed spaces from $(X, \mathcal{A}_X)$ to $(V, \mathcal{A}_V)$ for some (irreducible) real algebraic set V.

If $S \subset X$ is closed then S identifies with an algebraic set of $\mathbb{R}^n$ inside V. In this way, if S is irreducible, S acquires a canonical real algebraic affine variety structure; we call the corresponding structure sheaf $\mathcal{A}_{X,S}$.

A.4   Real Algebraic Varieties

In complete analogy with the complex case, real algebraic varieties are obtained by gluing together affine ones.

A *real algebraic variety* is a a pair $(X, \mathcal{A}_X)$ where X is a Noetherian irreducible topological space and $\mathcal{A}_X$ is a sheaf of $\mathbb{R}$-valued functions on X satisfying this condition: there exists a finite open cover $\{U_i\}_{i \in I}$ of X such that for each i, the ringed space $(U_i, \mathcal{A}_X|_{U_i})$ is a real affine algebraic variety. Then $\mathcal{A}_X$ is called the *structure sheaf* of X. The sections of $\mathcal{A}_X$ are the *regular functions* on U. The topology of X is then called the *Zariski topology*.

In speaking of real algebraic varieties, we may omit the modifiers "real" or "algebraic" when the context is clear. (However, often we will be dealing with complex algebraic varieties or real analytic manifolds at the same time.)

A regular mapping between varieties $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ is a Zariski continuous mapping $\phi : X \to Y$ such that if $U \subset Y$ is open and $f \in \mathcal{A}_Y(U)$ then $\phi^* f = f \circ \phi \in \mathcal{A}_X(\phi^{-1}U)$. An isomorphism is a bijective biregular map. We often speak of X as the variety and leave implicit its structure sheaf $\mathcal{A}_X$.

An *affine open* set U of X is then a Zariski open set U such that $(U, \mathcal{A}_X|_U)$ is an affine variety.

Let X be a a real algebraic variety Then we have the following examples of real algebraic subvarieties of X.

(i) Suppose W is Zariski open in X. Then W is again a variety where we define $\mathcal{A}_W$ by restriction of the structure sheaf of X.

(ii) Suppose S is an irreducible Zariski closed set in X. Then S is again a variety where for each open affine set $U \subset X$ we have $\mathcal{A}_S(S \cap U) = \mathcal{A}_{X,S}(S \cap U)$. If X is affine then so is S.

(iii) Now (i) and (ii) imply that any Zariski locally closed irreducible subset W of X is again a variety. For such a subvariety we may write $\mathcal{A}_X|_W$ for the induced structure sheaf

$\mathcal{A}_W$. A regular map $\phi : X \to Y$ of varieties is a *locally closed embedding* if $\phi(X)$ is a locally closed subvariety of Y and ϕ defines an isomorphism $(X, \mathcal{A}_X) \to (\phi(X), \mathcal{A}_Y|_{\phi(X)})$.

A.5   Real Structures and Real Forms

A complex algebraic variety Z is *defined over* $\mathbb{R}$ if Z is equipped with an involution

$$\kappa : Z \to Z$$

called complex conjugation, which satisfies the following: Z admits a cover by complex algebraic κ-stable affine open subsets U such that

(i) if $f \in R(U)$ then the function $\overline{f}$ defined by $\overline{f}(u) = \overline{f(\kappa(u))}$, $u \in U$, also lies in $R(U)$, and

(ii) the map $R(U) \to R(U)$, $f \mapsto \overline{f}$, is an $\mathbb{R}$-algebra involution of $R(U)$. In other words, the real subspace $\{f \in R(U) \,|\, f = \overline{f}\}$ is both a real form and a real subalgebra of $R(U)$.

We call this collection of real forms, or κ itself, a *real structure* on Z. We write $\overline{z} = \kappa(z)$ for $z \in Z$. Geometric objects on Z such as functions, vector fields and differential forms are *defined over* $\mathbb{R}$, or *real*, if they are stable under complex conjugation.

If $f : Z \to Z'$ is a complex algebraic morphism of complex varieties defined over $\mathbb{R}$, then f is *defined over* $\mathbb{R}$, or *real*, if f commutes with complex conjugation.

Let Z^κ be the set of real, i.e., κ-fixed points in Z. We put $Z(\mathbb{R}) = Z^\kappa$. Suppose $Z(\mathbb{R})$ is non-empty and Zariski irreducible. Then $Z(\mathbb{R})$ has a natural structure of real algebraic variety and we call $Z(\mathbb{R})$ a *real form* of Z.

To see this, we first define a Zariski topology on $Z(\mathbb{R})$ by the collection of sets

$$\mathcal{S} = \{U(\mathbb{R}) = U \cap Z(\mathbb{R}) \,|\, U \in \mathcal{T}\}$$

where $\mathcal{T}$ is the collection of κ-stable Zariski open subsets of Z. For each affine $W \in \mathcal{S}$ we define $\mathcal{A}_{Z(\mathbb{R})}(W)$ to be the space of quotients P/Q where P and Q are regular functions on some $U \in \mathcal{T}$ with $U(\mathbb{R}) = W$, P and Q real (i.e., κ-fixed) and Q nowhere vanishing on W. This data defines a unique sheaf $\mathcal{A}_{Z(\mathbb{R})}$ of $\mathbb{R}$-valued functions on $Z(\mathbb{R})$ and then $(Z(\mathbb{R}), \mathcal{A}_{Z(\mathbb{R})})$ is a real algebraic variety.

Notice that $Z(\mathbb{R})$ is affine if Z is affine. In fact suppose $Z \subset \mathbb{C}^n$ is defined by the vanishing of $P_1, \cdots, P_m \in \mathbb{C}[z_1, \ldots, z_n]$ and also Z is complex conjugation stable. Then $Z(\mathbb{R})$ is the zero-locus in $\mathbb{R}^n$ of the $2m$ real polynomial functions defined by the real and imaginary parts $\mathrm{Re}(P_1), \mathrm{Im}(P_1), \ldots, \mathrm{Re}(P_m), \mathrm{Im}(P_m) \in \mathbb{R}[z_1, \ldots, z_n]$.

Clearly every real affine algebraic variety is of the form $Z(\mathbb{R})$ for some complex affine algebraic variety Z defined over $\mathbb{R}$.

The process $Z \mapsto Z(\mathbb{R})$ is compatible with the usual operations on varieties. For instance, if $\phi : Z \to Z'$ is a regular map of complex algebraic varieties defined over $\mathbb{R}$, then the induced map $\phi(\mathbb{R}) : Z(\mathbb{R}) \to Z(\mathbb{R})'$ is a regular map of regular algebraic varieties.

Real structures often arise in the following way. Suppose V is a complex vector space and $V_\mathbb{R}$ is a real form of V with corresponding complex conjugation map $\kappa : V \to V$. Then κ defines a real structure on every κ-stable (locally closed) complex algebraic subvariety X of V.

Suppose a complex algebraic group H acts on Z and H is defined over $\mathbb{R}$. We say the H-*action on* Z *is defined over* $\mathbb{R}$ if the action morphism $H \times Z \to Z$ is defined over $\mathbb{R}$. This happens if and only if for every $h \in H$, the transformations of Z defined by h and $\overline{h}$ are complex conjugate.

A.6   The Complex Conjugate of a Complex Variety

Given a complex algebraic variety Z, we may construct another complex algebraic variety $\overline{Z}$ called the (abstract) *complex conjugate* variety. If Z is affine, then $\overline{Z}$ is the unique affine variety such that

$$R(\overline{Z}) = \overline{R(Z)}$$

where $\overline{R(Z)}$ is the $\mathbb{C}$-algebra which is complex conjugate to $R(Z)$; i.e., $\overline{R(Z)}$ has the same underlying $\mathbb{R}$-algebra structure but has the complex conjugate complex vector space structure. For general varieties, $\overline{Z}$ is defined in the obvious way by gluing together complex conjugate affine opens.

If $f : Z \rightarrow Z'$ is a morphism of complex varieties then the complex conjugate map $\overline{f} : \overline{Z} \rightarrow \overline{Z'}$ defined by

$$\overline{f}(p) = \overline{f(\overline{p})}$$

is also a morphism.

The construction of $\overline{Z}$ from Z is functorial in the usual ways and commutes with products. We have natural identifications $\overline{TZ} = T\overline{Z}$ and $\overline{T^*Z} = T^*\overline{Z}$ for the holomorphic tangent and cotangent bundles. Also pullback of differential forms and pushforward of vector fields commutes with taking the complex conjugate.

Consider the natural map

$$Z \rightarrow Z \times \overline{Z}, \qquad z \mapsto (z, \overline{z})$$

This embeds Z as a real form of $Z \times \overline{Z}$ with respect to the real structure defined by $\overline{(u,v)} = (\overline{v}, \overline{u})$. Thus in particular, Z itself has a canonical structure of real algebraic variety. This amounts to "forgetting" part of the complex algebraic variety structure. Notice that Z and $\overline{Z}$ acquire isomorphic real algebraic variety structures in this way.

We may write Z^{real} for Z regarded as real variety. If Z is an affine complex variety then Z^{real} is just the obvious affine real variety. Indeed suppose $Z \subset \mathbb{C}^n$ is defined by the vanishing of $P_1, \cdots, P_m \in \mathbb{C}[z_1, \ldots, z_n]$. We have a natural $\mathbb{R}$-algebra homomorphism $\mathbb{C}[z_1, \ldots, z_n] \rightarrow \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$, say $P \mapsto P'$, defined by setting $z'_j = x_j + iy_j$. Then $Z^{real} \subset \mathbb{R}^{2n}$ is the closed real algebraic subvariety defined by the vanishing of the real and imaginary parts $\operatorname{Re} P'_1, \operatorname{Im} P'_1, \ldots, \operatorname{Re} P'_m, \operatorname{Im} P'_m \in \mathbb{R}[x_1, y_1, \ldots, x_n, y_n]$.

If Z has a real structure κ, then the map

$$Z \rightarrow \overline{Z}, \qquad z \mapsto \overline{\kappa(z)}$$

is an isomorphism of complex algebraic varieties .

A.7   Tangent Spaces, Dimension, and Smoothness

Let v be a point of an irreducible algebraic set $V \subset \mathbb{R}^n$. The *Zariski tangent space T_vV at* v may be defined as the linear subspace of $\mathbb{R}^n$ given by

$$T_vV = \{x \in \mathbb{R}^n \,|\, (\operatorname{grad}P|_v) \cdot x = 0 \text{ for all } P \in I(V)\}$$

The dimension $d_v = \dim T_v V$ is generically the same over V (i.e., is the same over some Zariski open dense set of V). This common value of d_v is called the *dimension d_V* of V. A point $v \in V$ is a *smooth point* if $d_v = d_V$. The set V^{reg} of smooth points is Zariski open dense in V. V is a *smooth variety* if $V = V^{reg}$.

These notions pass immediately to affine real algebraic varieties and then are purely local. These notions then pass to general real algebraic varieties as the latter are obtained by gluing of affine opens. In particular then the notions of Zariski tangent space and smooth point are purely local. In the usual way one defines *étale maps* of real algebraic varieties.

If Z is smooth, then, in the context of §A.5, $Z(\mathbb{R})$ is a smooth real form of Z (in particular $Z(\mathbb{R})$ is irreducible). This follows by observing that at each point $z \in Z(\mathbb{R})$ the complex Zariski tangent space $T_z Z$ acquires a real structure and then the real points form the tangent space to the real submanifold $Z(\mathbb{R})$.

A smooth real algebraic variety X has a natural structure of real analytic manifold, just as a smooth complex algebraic variety has a natural structure of complex analytic manifold. In particular X has a larger topology, often called the *strong or Euclidean topology*, which refines the Zariski topology. On $\mathbb{R}^n$, this is just the usual Euclidean topology.

Now X, while connected in the Zariski topology (since it is irreducible), may well fail to be connected in the Euclidean topology. This typically happens when taking real forms. For example, the familiar real form of $\mathbb{C}^*$ is $\mathbb{R}^*$. Fortunately, the individual Euclidean connected components have a natural structure, namely each is a semi-algebraic real analytic submanifold. In fact, each component is a *Nash manifold*. We develop this notion in the rest of this Appendix. The starting point is semi-algebraic sets.

A.8   Real Semi-Algebraic Sets and Maps

A subset $S \subset \mathbb{R}^n$ is a real *semi-algebraic set* if S is a finite union of sets of the form:

$$\{x \in \mathbb{R}^n \mid P_1(x) = \cdots = P_m = 0 \text{ and } Q_1(x), \ldots, Q_m(x) > 0\}$$

where $P_i, Q_j \in \mathbb{R}[x_1, \ldots, x_n]$.

Suppose $S \subset \mathbb{R}^n$ and $T \subset \mathbb{R}^m$ are semi-algebraic sets. A map $\phi : S \to T$ is a *semi-algebraic map* if the graph of ϕ is a semi-algebraic set in $\mathbb{R}^{n+m}$. A semi-algebraic map $f : S \to \mathbb{R}$ is called a *semi-algebraic function*. It follows that ϕ is semi-algebraic if and only if all the component functions $\phi_1, \ldots, \phi_m$ are semi-algebraic.

Notice that a regular map of real algebraic sets in Euclidean space is in particular a semi-algebraic map of semi-algebraic sets.

Semi-algebraic sets arise inevitably in the study of real algebraic sets. Indeed the image of an algebraic set under a regular mapping, even a linear projection of Euclidean space, is in general only semi-algebraic. Also the connected components (in the Euclidean topology) of an algebraic set are generally only semi-algebraic.

The Tarski-Seidenberg Theorem says that under a semi-algebraic map, the image of a semi-algebraic set is semi-algebraic. Another result says that a semi-algebraic set has finitely many connected components (in the Euclidean topology) and each such component is semi-algebraic (see [BoCR, Th. 2.4.5, pg 31]).

Next we define semi-algebraic sets in varieties. Let $(X, \mathcal{A}_X)$ be a real algebraic variety and let $S \subset X$. If X is affine, then we call S *semi-algebraic* if for one (and hence every) closed embedding $\phi : X \to \mathbb{R}^n$ of real algebraic varieties, the set $\phi(S)$ is semi-algebraic in $\mathbb{R}^n$.

Now for general X we call S *semi-algebraic* if for every affine open $U \subset X$ (or equivalently, for every member U of some affine open cover of X), the set $S \cap U$ is semi-algebraic in U.

Notice that if $W \subset X$ is a locally closed subvariety, then $S \cap W$ is semi-algebraic in W.

Now, generalizing the definition above, if $S \subset X$ and $T \subset Y$ are semi-algebraic sets in real algebraic varieties, then a map $\phi : S \to T$ is *semi-algebraic* if the graph of ϕ is semi-algebraic in $X \times Y$. It is routine to check that the Tarski-Seidenberg theorem is still true in this setting.

A Euclidean open semi-algebraic set, and so in particular a Euclidean connected component, of a smooth real algebraic variety is a real analytic submanifold.

An easy, but important observation is the following: if X is a real algebraic variety and $\phi \in A(X)$ is such that ϕ takes both positive and negative values on X then the set

$$S = (\phi > 0) \subset X$$

is semi-algebraic in X (but not algebraic).

A.9   Nash Functions and Nash Manifolds

Suppose that S is a (Euclidean) open semi-algebraic set in a smooth irreducible algebraic set $V \subset \mathbb{R}^n$.

A real analytic function $f : S \to \mathbb{R}$ is called a *Nash function* if f satisfies the following two equivalent conditions:

(i) f is algebraic over the algebra $P(V)$ of polynomial functions and

(ii) f is semi-algebraic.

The Nash functions form a Noetherian $\mathbb{R}$-algebra $N_V(S)$ algebraic over $P(V)$, and furthermore $N_V(S)$ is integrally closed if S is Euclidean connected — see [BoE].

From now on assume, more generally, that S is a semi-algebraic real analytic smooth submanifold of a smooth real algebraic variety X.

If X is affine, then the definition above of Nash function on S and the equivalence of the two conditions go over immediately as soon as we replace (i) by the condition:

(i') f is algebraic over $A(X)$.

(In the case $X = V$, this is consistent with the previous definition as $A(V)$ is algebraic over $P(V)$) The Nash functions on S form a Noetherian $\mathbb{R}$-algebra $N_X(S)$ which is algebraic over $A(X)$ and, if S is Euclidean connected, integrally closed.

Now we can treat the case where X is not necessarily affine. A real analytic function $f : S \to \mathbb{R}$ is a *Nash function* if f satisfies the following two equivalent conditions:

(i) for each affine open $U \subset X$ (or equivalently, for every member U of some affine open cover of X), the restriction $f|_{S \cap U}$ is algebraic over $A(U)$ and

(ii) f is semi-algebraic.

It follows from the affine case that the Nash functions form an $A(X)$-algebra $N_X(S)$ which is integrally closed if S is Euclidean connected.

Next we define the *sheaf $\mathcal{N}_S$ of Nash functions on S*. We start with the Euclidean topology on S. The collection $\mathcal{F}_S$ of semi-algebraic Euclidean open sets in S is a basis of this topology (e.g, use small open balls). If $U \in \mathcal{F}_S$, then we define $\mathcal{N}_S(U) = N_X(U)$. This data determines uniquely the sheaf $\mathcal{N}_S$ of $\mathbb{R}$-algebras on S.

The pair $(S, \mathcal{N}_S)$ is then an example of a *Nash manifold*. We will not develop a more general theory of Nash manifolds here as these examples are sufficient for purposes of studying orbits of real algebraic groups, as explained in §A.10 below.

In particular, smooth real algebraic varieties are Nash manifolds and all real algebraic constructions on them or among them are Nash in the sense discussed below.

Notice that our constructions on S have nice functorial properties. For example, if $X \subset X'$ is a (locally closed) embedding of smooth real algebraic varieties, then X and X' define the same Nash manifold structure on S.

Now suppose $S' \subset S$ is such that S' is a semi-algebraic real analytic smooth submanifold of X. Then S' with its sheaf $\mathcal{N}_{S'}$ of Nash functions, is a *Nash submanifold* of S. In particular, each Euclidean connected component of S is an open Nash submanifold.

If $(S, \mathcal{N}_S)$ and $(T, \mathcal{N}_T)$ are two Nash manifolds, then a morphism of the ringed spaces is called a *Nash map* or a *Nash morphism*. Thus a map $\phi : S \to T$ is Nash if and only if for each Euclidean open set $V \subset T$, $\phi^{-1}(V)$ is Euclidean open in S and ϕ defines an algebra homomorphism $\phi^* : \mathcal{N}_T(V) \to \mathcal{N}_S(\phi^{-1}(V))$ by pullback of functions. A Nash map ϕ is a *Nash isomorphism* if ϕ is bijective and ϕ^{-1} is Nash.

A Nash map $\phi : S \to T$ is a *Nash embedding* if $\phi(S)$ is a Nash submanifold of T and the restricted map $\phi : S \to \phi(S)$ is a Nash isomorphism.

In the natural way, we define Nash Lie groups, Nash group actions, etc.

We can define in the obvious way Nash fibrations and Nash coverings of Nash manifolds. We note that local triviality in the étale topology on real algebraic varieties implies local triviality in the Euclidean topology.

Then in particular we get the notion of a Nash vector bundle over a Nash manifold and the space of Nash sections. If X is a Nash manifold then the tangent and cotangent bundles of X have natural Nash bundle structures. Consequently, for any tensor field η on X, such as a vector field, a differential form, a metric or a complex structure, we define η to be *Nash* if the corresponding section of the bundle $TX^{\otimes r} \otimes T^*X^{\otimes s}$ is Nash. This gives notions of Nash symplectic manifold, Nash Riemannian manifold, Nash complex manifold, Nash Kaehler manifold, Nash hyperkaehler manifold, etc.

If X is a totally real Nash submanifold of a smooth complex algebraic variety Z such that $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} Z$, then we say that Z is a *Nash complexification* of X. A stronger condition on X is that X is a Euclidean connected component of the fixed-point set Z^κ for some real structure κ on Z. Then we say also that X is a *real form* of Z. This extends our definition of real form from §A.5 .

A.10   Orbits of Real Algebraic Groups

We consider now real algebraic groups $G(\mathbb{R})$ that arise in the following way. Let G be a Zariski connected complex algebraic group defined over $\mathbb{R}$ with group $G(\mathbb{R})$ of real points. We assume as usual that G is a complex affine algebraic variety; then $G(\mathbb{R})$ is a real affine algebraic variety. For example, compact Lie groups arise in this way.

Now $G(\mathbb{R})$ is Zariski connected but in general not Euclidean connected. For instance if $G = GL(n, \mathbb{C})$, $n \geq 1$, then $G(\mathbb{R}) = GL(n, \mathbb{R})$ has two connected components, defined by the sign of the determinant. The Euclidean connected component $G_{\mathbb{R}}$ of $G(\mathbb{R})$ is a semi-algebraic set in $G(\mathbb{R})$. If G is semisimple and simply-connected, then $G_{\mathbb{R}} = G(\mathbb{R})$.

Suppose G acts morphically on an (irreducible) complex algebraic variety X, i.e., G acts on X and the action map $G \times X \to X$ is a morphism of complex algebraic varieties. If X and the action (i.e., the action morphism) are defined over $\mathbb{R}$ then $G(\mathbb{R})$ acts morphically on the (irreducible) set $X(\mathbb{R})$ of real points. Each Euclidean connected component $X_{\mathbb{R}}$ of $X(\mathbb{R})$

is a semi-algebraic set in $X(\mathbb{R})$.

Each G-orbit $G \cdot x$ on X is a smooth locally closed (irreducible) complex algebraic sub-variety of X. Hence, if $x \in X(\mathbb{R})$, the set of real points

$$(G \cdot x)(\mathbb{R}) = (G \cdot x) \cap X(\mathbb{R})$$

is a smooth locally closed (irreducible) real algebraic subvariety of $X(\mathbb{R})$, and hence is a finite union of Euclidean connected components of the same dimension. These components are then semi-algebraic sets and moreover are Nash submanifolds.

On the other hand, by the Tarski-Seidenberg Theorem, the orbits $G(\mathbb{R}) \cdot x$ and $G_{\mathbb{R}} \cdot x$ are semi-algebraic sets in $X(\mathbb{R})$. In particular $G_{\mathbb{R}} \cdot x$ is a component of $(G \cdot x)(\mathbb{R})$.

Thus Nash manifolds are the natural objects in this setting. Finally we give an example of how Nash isomorphisms can arise. Consider the standard action of $G = S0(3, \mathbb{C})$ on $\mathbb{C}^3$ as the special orthogonal group of the quadratic form $x^2 + y^2 - z^2$ where x, y, z are real linear coordinates. The subset

$$X = (x^2 + y^2 = z^2) - \{(0, 0, 0)\}$$

is a G-orbit.

But $X(\mathbb{R})$ has two Euclidean connected components defined by the sign of z. Let $X_{\mathbb{R}}$ be the component where $z > 0$. The projection

$$p : X(\mathbb{R}) \to \mathbb{C} - \{0\}, \qquad p(x, y, z) = x + iy$$

is a 2-to-1 étale real algebraic morphism. The restricted map $p_{\mathbb{R}} : X_{\mathbb{R}} \to \mathbb{C} - \{0\}$ defined by p is a Nash isomorphism. Indeed the inverse map is

$$\mathbb{C} - \{0\} \to X_{\mathbb{R}}, \qquad x + iy \mapsto \left(x, y, \sqrt{x^2 + y^2} \right)$$

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